

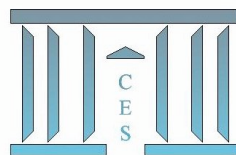


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Time, Bifurcations and Economic Applications

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2009.28



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(first draft: April 2, 2007; revised version: April 23, 2009)

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*This paper has been presented to the *2ème Journée Eco-Math* and to the workshop *Market Imperfections and Macroeconomic Instability* held at the University of Lille 1 on April 2, 2007, and on March 14-15, 2008, respectively; and to the ASSET conference held in Florence on November 7-9, 2008. We would like to thank all the participants to these meetings and, in particular, Kirill Borissov and Alain Ayong Le Kama for helpful comments.

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Abstract

In this paper, we apply first and higher-order Euler discretizations to compare dynamic systems in discrete and continuous time. In addition, we stress the difference between backward and forward-looking approximations. Focussing on local bifurcations, we find that time representation is neutral and asymptotically neutral for models with saddle-node and Hopf bifurcations, respectively. Conversely, it is far from neutral for models with flip bifurcations (in discrete time), even though these bifurcations disappear under a critical discretization step or under higher-order Euler discretizations.

In the second part, we apply the theory to popular economic models. Discrete-time dynamics of capital accumulation, such as Solow (1956), can be recovered under first-order backward-looking discretizations because of the predetermined nature of capital. Models of capital accumulation with intertemporal optimization, such as Ramsey (1928), need hybrid discretizations because of the forward-looking nature of the Euler equation, where consumption behaves as jumping variable.

Résumé

Dans ce papier, nous effectuons une comparaison des systèmes dynamiques en temps discret et en temps continu fondée sur des discrétisations d'Euler du premier ordre et d'ordre supérieur. Nous montrons que l'équivalence entre le modèle discret et la version discrétisée du modèle continu peut être obtenue généralement par une approximation tournée vers le passé (*backward-looking*), une approximation tournée vers le futur (*forward-looking*) et dans certains cas par une approximation hybride. Pour une analyse qui se limite aux bifurcations locales élémentaires, on trouve que la représentation du temps est neutre dans les modèles avec bifurcations de type *saddle-node* et asymptotiquement neutre pour des bifurcations Hopf. Par contre, ce résultat de neutralité ne tient plus en présence de bifurcations *flip* (en temps discret), même si ces bifurcations disparaissent en dessous d'une valeur critique du pas de discrétisation ou au dessus d'un certain ordre de discrétisation.

Dans une deuxième partie du papier, nous illustrons nos résultats théoriques en appliquant la méthodologie à un certain nombre de modèles économiques familiers. On retrouve la dynamique d'accumulation du capital (variable prédéterminée) à la Solow (1956) en temps discret, à partir d'une discrétisation au premier ordre *backward-looking* de sa version en temps continu. Dans les modèles de croissance à la Ramsey (1928), avec optimisation intertemporelle, cette équivalence entre temps discret et temps continu ne peut provenir que d'une discrétisation hybride : à cette approximation *backward-looking* de la dynamique du capital doit s'adjoindre une discrétisation *forward-looking* de la dynamique de la consommation en raison de la nature tournée vers le futur de l'équation d'Euler.

Keywords: discretizations, bifurcations, growth models.

JEL Classification: C02, C61, C62, O41.

1 Introduction

The issue of time representation, that is, the choice of a discrete or a continuous variable, is a fundamental concern in economic theory. Turnovsky (1977) wrote in *International Macroeconomic Dynamics*: "There is one methodological matter that arises in the modeling of dynamic economic systems, and that is the choice of discrete versus continuous time [...] The preference in this book is for continuous time formulation, mainly because we find it to be more tractable and often more transparent. But to some degree this choice is a question of taste."

In our paper, we don't address the question whether discrete or continuous-time models are more appropriate to represent the economic activity. It is deceptive to answer such a question; there are neither criteria nor measures to compare the suitability of these approaches. In this respect, Turnovsky is right: it's a matter of taste.

Optimization in discrete (continuous) time and the solution of difference (differential) equations often rests on the author's skills and mathematical convenience. In most of cases, authors are forced to a given option by no other reason than formal easiness.

We aim at drawing the aftermath of time structure on the economic system without questioning on the reasons before. More precisely, the question we raise is how the choice of time, as a continuous or discrete variable, affects the stability properties of a dynamic system: it is well-known that these properties can depend on the time specification.

Most of theoretical models, especially in the growth literature, are built in continuous time. Preference for system of differential equations comes essentially from technical considerations. Continuous-time systems are usually more tractable than systems of difference equations. If, on the one hand, most of economic transactions are pointwise over time, on the other hand, the common sense suggests that real life unfolds continuously. More sophisticated economic arguments are developed by Gandolfo (1997).¹

On the one side, decisions and transactions happen at given instants and statistical data are available as discrete-time measurements. Some authors argue that a discrete-time approach makes sense not only from a theoretical but also from an empirical point of view.²

From a methodological point of view, there is another difference between these representations which argues in favour of discrete time. A simple one-dimensional difference equation can generate complex dynamics, while a higher-dimensional system is needed in continuous time (Guckenheimer and Holmes (1983)).

¹Gandolfo (1997) puts forward eight arguments in favour of a continuous instead of a discrete-time representation of economic activity.

²Two main criticisms are addressed by Gandolfo (1997) to these apparently convincing arguments. First, although individual decisions are discrete, the fact that they are not synchronized and spread over time from a great number of agents, restores a theoretical justification for continuous-time models. In addition, statistical inference in continuous time knew consequent and satisfactory developments since the 1970s (see Bergstrom (1976), Bergstrom (1984), Gandolfo (1981) and Wymer (1972)).

For example, the logistic map exhibits stable fixed point, stable periodic cycles (of any order) and deterministic chaos. In addition, all these dynamic behaviors are sensitive to a single parameter value. Conversely, only monotonic orbits, either convergent or explosive, are generated by a one-dimensional differential equation. Consequently, one gains in simplicity by modeling complex dynamics in discrete time.

Finally, in discrete time, distinction between forward and backward-looking variables turns out to be more natural. For instance, introducing observed or expected inflation in a Taylor rule changes the dynamic properties of monetary policy.

These examples show that time modeling is neither trivial nor neutral and has economic consequences. The choice of time can determine the results independently of the underlying economic mechanisms. In the logistic case, the continuous time rules out in advance the occurrence of (a)periodic cycles.

A still new but growing literature is focussing on a common ground to both the strands of models and checking the sensitivity of results to time representation. Scholars tackle the question in different ways.

- Mercenier and Michel (1994) consider infinite-horizon optimizations and discretize continuous-time models as usually done in numerical simulation. Their goal is closely related to ours. A discrete-time approximation should, at least, preserve the steady-state of the continuous-time model. They show that the invariance property of steady state can be achieved through an appropriate Euler discretization and simple restrictions on discounting.
- Some papers reconsider in discrete time the dynamic properties of a class of models originally written in continuous time (and *vice versa*).

A strand of literature focuses on dynamic indeterminacy. Carlstrom and Fuerst (2005) study the role of time specification on real indeterminacy and sunspot equilibria in models where the central bank implements a fairly interest rate policy rule. Recommendations to rule out multiple equilibria are substantially different under different time representations. Mino, Nishimura, Shimomura and Wang (2005) address the same issue in two-sector endogenous growth models with constant social returns. As Mercenier and Michel (1994), they are concerned with a discrepancy of results in theoretical continuous-time works and discrete-time applied papers. In the class of endogenous growth models they consider, they find that conventional results of continuous-time literature no longer hold in a discrete-time formulation. Hintermaier (2005) shows that the existence of sunspot equilibria in business cycle models depends on the specification of time and, in discrete time, on the period length.

Another popular class of models, time-to-build, has been recently revisited in the light of time specification. Using delay equations (difference-differential equations) seems particularly appropriate and somewhat appealing to tackle our issue. Licandro and Puch (2006) study whether the discrete-time version is consistent with the continuous-time time-to-build

model. Bambi (2008) makes an attempt to unify the literature on time-to-build, recovering multi-period investments in discrete-time from delay equations in continuous time. Cycles are found to occur (through Hopf bifurcations) under both time representations.

Eventually, one may question whether time nature matters under uncertainty. Leung (1995) shows that the consumption paths are different in discrete and continuous time when agents face an uncertain life-span.

- Anagnostopoulos and Giannitsarou (2008) build a dynamic general equilibrium model where the period length is a free parameter. They recover traditional models in discrete and continuous time as particular cases. In addition, they compare the dynamic properties of a large class of models under both time specifications. Focussing on indeterminacy, they conclude that the period length matters in economic dynamics.

Comparing conditions for the occurrence of elementary bifurcations in a neighborhood of an invariant steady state is a proper way of tackling the question. Prior to bifurcation analysis, we need a general approach in the spirit of Anagnostopoulos and Giannitsarou (2008) and we bridge the continuous and discrete-time models through a general discretization method. In this respect, we apply, as Mercenier and Michel (1994) and Krivine, Lesne and Treiner (2007), an Euler procedure based on a Taylor expansion to discretize a system of differential equations. First or higher-order Euler discretizations preserve the continuous-time steady state and help us to understand why some stability properties (dis)appear from a time representation to another. An appealing feature of Euler approximations is that discrete-time dynamics can be recovered under opportune discretizations. More precisely, while the steady state is invariant under discretization, whatever its order and step, dynamic properties are sensitive to both the order and the step. In addition, they depend on the type of discretization (backward, forward-looking or hybrid). Nevertheless, discrete-time dynamics of popular growth models can be derived from the continuous-time system through first-order unit-step discretizations. Focussing on the type of discretization is a value we add to the existing literature. In economics, some models require backward or forward-looking discretizations (Solow (1956)), while a large class of higher-dimensional models need hybrid discretization to recover the equivalence between discrete and continuous time. The mix of backward and forward-looking discretizations, what we call hybrid, is particularly appropriate for models of dynamic optimization. Traditional optimal growth models in discrete time (Ramsey (1928), Cass-Koopmans (1965)) come from a hybrid approximation of the continuous-time versions: backward-looking discretization of the law of motion involving the state variable (as in Solow) and forward-looking discretization of the Euler equation involving a costate variable (the discounted multiplier).

Bifurcation theory is a fashionable topic built on a solid ground by generations of brilliant mathematicians (from Poincaré to Thom, to Arnol'd). A literature within the reach of nonspecialized scholars has grown in recent years.

Good introductions are, among the others, Guckenheimer and Holmes (1983), Hale and Koçak (1991). However, in order to have a rigorous but concise introduction to bifurcations in discrete time, interested scholars are highly recommended to read Grandmont (2008).

In the first part of the paper, we provide a comparative analysis of (local) bifurcations in continuous and discrete time bridging the models through polynomial approximations and exploiting the equivalence properties of discrete-time and (opportunistically) discretized systems: bifurcations depend on the discretization step. At the best of our knowledge, no attempts have been made in literature to tackle the issue from this point of view. In addition, we go beyond the equivalence results and we study also the occurrence of bifurcations in higher-order discretizations, including quadratic forms.

For simplicity, we focus on two-dimensional systems. The Central Manifold Theorem ensures that one or two-dimensional manifolds (in the case of Hopf bifurcations) are concerned by generic bifurcations.

We consider three classes of elementary bifurcations: the family of saddle node bifurcations (including the transcritical and the pitchfork), the family of period-doubling bifurcations (including the flip), the family of Hopf bifurcations.

The dynamic properties in continuous time of a saddle-node bifurcation are preserved under a Euler discretization, whatever the discretization step.

From a qualitative point of view, Hopf bifurcations are also preserved, but no longer for arbitrary discretization steps. More precisely, a "continuity" property holds under discretization: conditions for Hopf bifurcation in discrete time "tend" to conditions in continuous time as the discretization step tends to zero.

The very difference arises in the case of flip bifurcations that are specific to discrete-time systems (indeed, there is no room for period-doubling bifurcations in continuous time). Under linear Euler approximations, the flip bifurcation disappears when the discretization step falls below a positive critical value. In addition, we show a surprising result in traditional growth models: a quadratic Euler approximation rules out the occurrence of flip whatever the discretization step.

We can say that the choice of time representation is neutral for models with saddle node bifurcations, almost neutral for models with Hopf bifurcations (the discrete-time critical value lies in a neighborhood of the continuous-time bifurcation value), far from neutral for models with flip bifurcations (in discrete time), even though these bifurcations disappear below a given threshold.

In the second part of the paper, we apply the methodology to popular dynamic economic models. We start with models, such as Kaldor (1940) or Solow (1956), where the discrete-time counterpart of continuous-time system comes from backward-looking discretizations.

In the seminal Solow's dynamics, there is no room for bifurcations: the steady state is stable whatever the time representation. In addition, we show that computing the intensive form of discretized dynamics is not equivalent to discretizing the intensive form of the original model, even though the steady state remains the same. In the second case, flip bifurcations require a sufficiently large discretization step. As seen above, a second (and *a fortiori* higher)

order Euler discretization makes the discretized system sufficiently close to the continuous-time model to rule out any bifurcation, whatever the discretization step. As usual, introducing imperfections, like externalities, can promote non-monotonic dynamics. Following Day (1982), we add a negative productive externality in the Solow model: there is room for flip bifurcations in discrete time and, unsurprisingly, no room for bifurcations in continuous time. The analysis of the discretized model allows us to compute the critical discretization step.

In order to study the occurrence of Hopf bifurcations, we need two-dimensional dynamics. Kaldor model is one of the simplest economic example of two-dimensional systems: a Hopf bifurcation arises under a parsimonious set of assumptions. Confirming the general results found in the first part, we show that, even if the discrete and the continuous-time critical values differ, the former converges to the latter as the discretization step tends to zero.

Applications carry on with an influential class of optimal growth models. It's a question of models with dynamic optimization, namely intertemporal utility maximization. The original Ramsey (1928), with undiscounted preferences, is a good introduction. We show that an appropriate hybrid discretization is needed to recover the discrete-time from the continuous-time system. The general hybrid approximation developed in the first methodological part can be now applied to the reduced forms of the Ramsey model.

In 1965, Cass and Koopmans introduced (independently) a discounting mechanism in the Ramsey model. Applying a Euler discretization to these models allows us to penetrate the sense of discounting in discrete/continuous time. Indeed, only discretizing the version of Euler equation with the discounted multiplier enables us to recover the traditional discrete-time model. The undiscounted multiplier is stationary at the steady state. Discretizing the Euler equation with the undiscounted multiplier gives an alternative approximation, which is still right but different from the usual intertemporal arbitrage in discrete time. There is no room for bifurcations in this model: the steady state is saddle-point stable in continuous as well as in discrete time, whatever the discretization step. To conclude the economic applications (the second part of the paper) and give a straightforward example of Hopf bifurcation in a model *à la* Cass-Koopmans with public spending externalities both in the utility and the production functions, we provide a more general³ version of Zhang (2000) and we show that both the externalities are crucial to generate limit cycles (through a Hopf bifurcation). We confirm the qualitative equivalence of the Hopf bifurcation highlighted in the first part of the paper: under a sufficiently small discretization step, this bifurcation occurs in discrete time if and only if it arises in continuous time.

The rest of the paper is organized as follows.

In the first part, we address the methodological issue of time discretization and the consequences on a dynamic system: the discretization methodology and the taxonomy of elementary bifurcations are presented in sections 2 and 3, respectively.

³Because of more general fundamentals.

The second part is devoted to apply the discretization methodology and the bifurcation analysis to some popular economic models. Section 4 focuses on backward-looking discretizations with applications to one-dimensional Solow models (flip bifurcation) and two-dimensional Kaldor models (Hopf bifurcation). In section 5, we apply the methodology to models of dynamic optimization, including seminal models of optimal growth, where the discrete-time version can be recovered under an opportune hybrid discretization.

Part I

Theory

2 General methodology

Time is discrete when is represented by a countable set of points. Time is continuous when is dense in the real numbers. In the first case, there exists a bijection with the set of natural numbers; in the second case with the continuum, say the real line.

Numerical methods have been applied to solve systems of differential equations. Discretizing, that is picking a sequence of points in the continuum, is a convenient way to represent and plot continuous-time system. Discretizations based on a Taylor development (that is polynomial representations) were introduced by Euler and are today quite pervasive in computational science.

From a theoretical point of view, the Euler approach can shed a light on the interplay between continuous and discrete-time dynamics. More precisely, it proves to be also pertinent to investigate and compare bifurcations in different timings.

The issue we tackle concerns the type and the amplitude of discretization we need in order to recover an equivalence between discrete and continuous time, where equivalence means the persistence of dynamic properties from a regime to another.

As Euler, we choose to apply a Taylor expansion to discretize a continuous-time system. We start by taking in account a first-order expansion (that is a linear discretization), then, we will consider higher-order discretizations.

2.1 First-order Euler discretization

2.1.1 Backward-looking discretizations

Instead of considering a continuous variable t and the corresponding position $x(t)$ determined by a system of ordinary differential equations:

$$\dot{x} = f(x) \tag{1}$$

jointly with the initial condition $x_0 \equiv x(0)$, let us pick up a regular sequence of time values: $(t_n)_{n=0}^{\infty} = (nh)_{n=0}^{\infty}$, where h is a (possibly small) positive constant

(discretization step), and the associated values: $x_n \equiv x(t_n) = x(nh)$.

According to the definition of derivative, we can write:

$$\dot{x}(t) \equiv \begin{bmatrix} \lim_{h \rightarrow 0} [x_1(t+h) - x_1(t)]/h \\ \vdots \\ \lim_{h \rightarrow 0} [x_m(t+h) - x_m(t)]/h \end{bmatrix}$$

where $x \in R^m$.

If h is sufficiently small, we can set $\dot{x}(t) \approx [x(t+h) - x(t)]/h$ and, therefore, $[x(t+h) - x(t)]/h \approx f(x(t))$. We obtain $[x(t_n+h) - x(t_n)]/h \approx f(x(t_n))$, that is $[x(t_{n+1}) - x(t_n)]/h \approx f(x(t_n))$ and, finally, $(x_{n+1} - x_n)/h \approx f(x_n)$, since $t_{n+1} = t_n + h$. It follows that

$$x_{n+1} - x_n \approx hf(x_n) \quad (2)$$

The entire sequence $(x_n)_{n=0}^\infty$ can be computed from the initial condition x_0 , by iterating the procedure: $x_1 \approx x_0 + hf(x_0)$, $x_2 \approx x_1 + hf(x_1) \approx x_1 + hf(x_0 + hf(x_0))$ and so on.

Equation (2) constitutes a backward-looking Euler discretization, because the variation $x_{n+1} - x_n$ depends on the past value x_n on the RHS of (2).

However, the sequence (x_n) is just an approximation of the true sequence $(x(nh))$, exact solution of system (1): the smaller h , the more accurate the approximation. It is not unworthy to observe that simplicity and elegance make the Euler's method the easiest technique to handle in order to plot a phase diagram and find a numerical solution of a system of differential equations.

However, we are not interested here in numerical simulations, but in the change of dynamic properties, when one passes from a continuous to a discrete time setting: Euler's discretization is of great help to understand why some stability properties (dis)appear from a timing to another.

Conversely, given an ordinary m -dimensional discrete-time system: $x_{n+1} = x_n + hf(x_n) \equiv g(x_n)$, we can define $f(x_n) \equiv [g(x_n) - x_n]/h$ and approximate the discrete-time system with $\dot{x} = f(x)$. As above, the smaller h , the more accurate the approximation.

2.1.2 Forward-looking discretizations

Instead of considering the right-hand derivative, we can compute the left-hand derivative: $\dot{x}(t) \equiv \lim_{h \rightarrow 0} ([x(t) - x(t-h)]/h)$ where $x \in R^m$.

If h is sufficiently small, we can set $\dot{x}(t) \approx [x(t) - x(t-h)]/h$ and, therefore, $[x(t) - x(t-h)]/h \approx f(x(t))$. We obtain $[x(t_{n+1}) - x(t_{n+1}-h)]/h \approx f(x(t_{n+1}))$, that is $[x(t_{n+1}) - x(t_n)]/h \approx f(x(t_{n+1}))$ and, finally, $(x_{n+1} - x_n)/h \approx f(x_{n+1})$, since $t_n = t_{n+1} - h$. It follows that

$$x_{n+1} - x_n \approx hf(x_{n+1}) \quad (3)$$

(3) constitutes a forward-looking Euler discretization, because the variation $x_{n+1} - x_n$ depends on the future value x_{n+1} on the RHS of (3).

In the following, we focus on higher-order Euler discretizations. As in the case of equation (3), the derivation of higher-order forward-looking discretizations is similar to that of backward-looking discretizations.

In economics, forward-looking discretizations are of interest because agents behave according to their expectations. In the second part of the paper, we will revisit influential economic models through hybrid linear approximations mixing backward and forward-looking discretizations.

2.2 Higher-order Euler discretizations

Let us now provide higher-order Taylor expansions to approximate the continuous-time dynamic system.

First, we begin with studying quadratic and higher-order polynomial approximations of a simple differential equation, then we will approximate higher-dimensional ordinary system of differential equations.

2.2.1 One-dimensional dynamics

Before entering higher-order discretizations of a dynamic system, let us show how to improve the Euler approximation of a simple one-dimensional dynamics through a quadratic Taylor expansion.

Proposition 1 *The differential equation $\dot{x} = f(x)$, $x \in R$, can be approximated by a second-order Taylor polynomial:*

$$x_{n+1} \approx x_n + f(x_n)h + f(x_n)f'(x_n)h^2/2 \equiv g(x_n) \quad (4)$$

Proof. As seen above, we define $x_n \equiv x(t_n) = x(nh)$. Then

$$x_{n+1} - x_n = x(nh + h) - x(nh) = \int_{nh}^{nh+\tau} \dot{x} dt \Big|_{\tau=h} = \int_{nh}^{nh+\tau} f(x(t)) dt \Big|_{\tau=h}$$

First, we compute

$$\varphi(\tau) \equiv \int_{nh}^{nh+\tau} f(x(t)) dt \quad (5)$$

through a (quadratic) Taylor expansion centered in $\tau = 0$.

$$\begin{aligned} \varphi(\tau) &\approx \varphi(0) + \varphi'(0)(\tau - 0) + \varphi''(0)(\tau - 0)^2/2 \\ &= \int_{nh}^{nh+\tau} f(x(t)) dt + f(x(nh+0))\tau + f'(x(nh+0))x'(nh+0)\tau^2/2 \\ &= 0 + f(x(nh))\tau + f'(x(nh))f(x(nh))\tau^2/2 \\ &= f(x_n)\tau + f'(x_n)f(x_n)\tau^2/2 \end{aligned}$$

since $x'(nh) = f(x(nh))$. Eventually, we obtain

$$x_{n+1} - x_n = \int_{nh}^{nh+h} f(x(t)) dt = \varphi(h) \approx f(x_n)h + f(x_n)f'(x_n)h^2/2$$

■

Similarly, one can compute a third-order Euler discretization:

$$\begin{aligned} x_{n+1} - x_n &\approx f(x_n)h + f(x_n)f'(x_n)h^2/2 + \left[f(x_n)f'(x_n)^2 + f(x_n)^2 f''(x_n) \right] h^3/6 \end{aligned}$$

If $f \in C^\infty$, the infinite-order Taylor series converges exactly to $x_{n+1} - x_n$ and one can replace the sign of approximation: \approx , with the sign of equality: $=$.

In other terms, if $f \in C^\infty$, an infinite-order discretization (Taylor series instead of a Taylor polynomial) makes the discrete-time dynamics exactly equal to those in continuous time. In this case, discretization introduces no distortions whatever the step h :

$$x_{n+1} - x_n = \varphi(h) = \sum_{p=0}^{\infty} \varphi^{(p)}(0) h^p / p! \quad (6)$$

where $\varphi^{(p)}$ denotes the p th derivative of function φ (see formula (5)).

A higher-order approximation requires numerical computations because of the infinite complexity of any analytical solution.

2.2.2 Higher-dimensional dynamics

System $\dot{x} = f(x)$ and its first-order approximation $x_{n+1} \approx x_n + hf(x_n)$ become, respectively:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_m) \\ &\vdots \\ \dot{x}_m &= f_m(x_1, \dots, x_m) \end{aligned}$$

and

$$\begin{aligned} x_{1n+1} &\approx x_{1n} + hf_1(x_{1n}, \dots, x_{mn}) \equiv g_1(x_{1n}, \dots, x_{mn}) \\ &\vdots \\ x_{mn+1} &\approx x_{mn} + hf_m(x_{1n}, \dots, x_{mn}) \equiv g_m(x_{1n}, \dots, x_{mn}) \end{aligned}$$

What could we say about the second-order approximation in this general case? As seen above, we define $x_n \equiv x(t_n) = x(nh)$ (in vector terms). More

explicitly, we have

$$\begin{aligned} x_{1n} &\equiv x_1(t_n) = x_1(nh) \\ &\vdots \\ x_{mn} &\equiv x_m(t_n) = x_m(nh) \end{aligned}$$

The following proposition holds.

Proposition 2 *The continuous-time dynamic system $\dot{x} = f(x)$, $x \in R^m$, can be approximated by a second-order Taylor polynomial:*

$$x_{in+1} \approx x_{in} + hf_i(x_n) + \frac{h^2}{2} \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{jn}}(x_n) \quad (7)$$

where the subscript i denotes the i th component of the vector.

Proof. Focus on the i th component. As above

$$x_{in+1} - x_{in} = x_i(nh + h) - x_i(nh) = \int_{nh}^{nh+h} \dot{x}_i dt \Big|_{\tau=nh}^{nh+h} = \int_{nh}^{nh+h} f_i(x(t)) dt \Big|_{\tau=nh}^{nh+h} \quad (8)$$

(notice that now f_i depends on all the components, that is, on the vector $x(t)$).

First, we compute

$$\varphi_i(\tau) \equiv \int_{nh}^{nh+\tau} f_i(x(t)) dt \quad (9)$$

through a (quadratic) Taylor expansion centered in $\tau = 0$: $\varphi_i(\tau) \approx \varphi_i(0) + \varphi_i'(0)(\tau - 0) + \varphi_i''(0)(\tau - 0)^2/2$. Explicitly, we get $\varphi_i(0) = \int_{nh}^{nh} f_i(x(t)) dt = 0$ and $\varphi_i'(0) = f_i(x(nh + 0)) = f_i(x_n)$. Moreover,

$$\begin{aligned} \varphi_i''(\tau) &= \frac{d}{d\tau} f_i(x(nh + \tau)) = \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(x(nh + \tau)) \frac{dx_j}{d\tau}(nh + \tau) \\ &= \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(x(nh + \tau)) f_j(x(nh + \tau)) \end{aligned}$$

and

$$\varphi_i''(0) = \sum_{j=1}^m f_j(x(nh)) \frac{\partial f_i}{\partial x_j}(x(nh)) = \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{jn}}(x_n)$$

since, with some notational misuse, $(dx_j/d\tau)(nh + \tau) = f_j(x(nh + \tau))$. Therefore, the i th component of φ becomes

$$\varphi_i(\tau) \approx \varphi_i(0) + \tau \varphi_i'(0) + \frac{\tau^2}{2} \varphi_i''(0) = \tau f_i(x_n) + \frac{\tau^2}{2} \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{jn}}(x_n)$$

Finally, according to (8) and (9), we obtain

$$x_{in+1} - x_{in} = \varphi_i(h) \approx h f_i(x_n) + \frac{h^2}{2} \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{j_n}}(x_n)$$

or, equivalently, (7). ■

In the case of a two-dimensional dynamics, the backward-looking formula (7) becomes explicitly:

$$\begin{bmatrix} x_{1n+1} - x_{1n} \\ x_{2n+1} - x_{2n} \end{bmatrix} \approx h \begin{bmatrix} f_1(x_{1n}, x_{2n}) \\ f_2(x_{1n}, x_{2n}) \end{bmatrix} + \frac{h^2}{2} \begin{bmatrix} \partial f_1 / \partial x_{1n} & \partial f_1 / \partial x_{2n} \\ \partial f_2 / \partial x_{1n} & \partial f_2 / \partial x_{2n} \end{bmatrix} \begin{bmatrix} f_1(x_{1n}, x_{2n}) \\ f_2(x_{1n}, x_{2n}) \end{bmatrix} \quad (10)$$

or, more compactly,

$$x_{n+1} \approx x_n + \left[hI + \frac{h^2}{2} J_0(x_n) \right] f(x_n) \quad (11)$$

As in the linear case of equation (3), we can derive a quadratic forward-

looking approximation. Formally, we get⁴

$$x_{in+1} \approx x_{in} + hf_i(x_{n+1}) - \frac{h^2}{2} \sum_{j=1}^m f_j(x_{n+1}) \frac{\partial f_i}{\partial x_{jn+1}}(x_{n+1}) \quad (14)$$

Computing higher-order Taylor expansions is still possible, exploiting the properties of vector function φ and its relation with the original function f . More precisely, we have

$$x_{in+1} - x_{in} = \left. \int_{nh}^{nh+\tau} f_i(x(t)) dt \right|_{\tau=h} = \varphi_i(h) \approx \sum_{p=0}^q \frac{h^p}{p!} \varphi_i^{(p)}(0) \quad (15)$$

where the right-hand side is the q th-order approximation of the i th component of the vector $x_{n+1} - x_n$.

We observe that the p th derivative of the scalar function φ_i involves the $(p-1)$ th (simple and cross) derivatives of the vector function f : even when

⁴Focussing on the i th component, we find

$$x_{in} - x_{in-1} = x_i(nh) - x_i(nh-h) = \left. \int_{nh-h}^{nh} \dot{x}_i dt \right|_{\tau=h} = \left. \int_{nh-h}^{nh} f_i(x(t)) dt \right|_{\tau=h} \quad (12)$$

(notice that now f_i depends on all the components, that is, on the vector $x(t)$).

First, we compute

$$\varphi_i(\tau) \equiv \int_{nh-\tau}^{nh} f_i(x(t)) dt \quad (13)$$

through a (quadratic) Taylor expansion centered in $\tau = 0$: $\varphi_i(\tau) \approx \varphi_i(0) + \varphi_i'(0)(\tau - 0) + \varphi_i''(0)(\tau - 0)^2/2$. More explicitly, we get $\varphi_i(0) = \int_{nh}^{nh} f_i(x(t)) dt = 0$ and $\varphi_i'(0) = f_i(x(nh - 0)) = f_i(x_n)$. Moreover

$$\varphi_i''(\tau) = \frac{d}{d\tau} f_i(x(nh - \tau)) = - \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(x(nh - \tau)) f_j(x(nh - \tau))$$

and

$$\varphi_i''(0) = - \sum_{j=1}^m f_j(x(nh)) \frac{\partial f_i}{\partial x_j}(x(nh)) = - \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{jn}}(x_n)$$

since $(dx_j/d\tau)(nh - \tau) = -f_j(x(nh - \tau))$. Therefore, the i th component of φ becomes

$$\varphi_i(\tau) \approx \varphi_i(0) + \tau \varphi_i'(0) + \frac{\tau^2}{2} \varphi_i''(0) = \tau f_i(x_n) - \frac{\tau^2}{2} \sum_{j=1}^m f_j(x_n) \frac{\partial f_i}{\partial x_{jn}}(x_n)$$

Finally, according to (12) and (13), we obtain

$$x_{in+1} - x_{in} \approx hf_i(x_{n+1}) - \frac{h^2}{2} \sum_{j=1}^m f_j(x_{n+1}) \frac{\partial f_i}{\partial x_{jn+1}}(x_{n+1})$$

or, equivalently, (14).

computations turn out to be hard from an analytical point of view, they are often feasible from a numerical point of view. As above, the identity

$$x_{in+1} - x_{in} = \varphi_i(h) = \sum_{p=0}^{\infty} \frac{h^p}{p!} \varphi_i^{(p)}(0)$$

holds whatever h if $f \in C^\infty$.

3 Local bifurcations

In order to compare continuous-time and discrete-time system, we will study approximations in a neighborhood of the steady state and focus only on elementary bifurcations.

3.1 Steady state

The system $\dot{x} = f(x)$ and its discrete time approximation $x_{n+1} \approx x_n + hf(x_n)$ have the same steady state. Indeed, in the continuous-time case, we require $\dot{x} = 0$, that is $f(x) = 0$, while, in the discrete-time case, we need $x_{n+1} = x_n$, that is $f(x_n) = 0$ for every n , or, equivalently, $f(x) = 0$. More explicitly: $f_1(x_1, \dots, x_m) = \dots = f_m(x_1, \dots, x_m) = 0$ determines the steady state of either a continuous-time or a discrete-time dynamics. We further notice that the steady state equations $f_1(x_1, \dots, x_m) = \dots = f_m(x_1, \dots, x_m) = 0$ neither depend on the discretization degree h nor on the discretization method (forward-looking or backward-looking).

3.2 Elementary bifurcations

We consider (local) bifurcations in stability of a simple attractor: the steady state, and we study the role of either the order or the degree h of discretization in the occurrence of these bifurcations.

In continuous time, an elementary bifurcation arises when the real part of an eigenvalue $\lambda(p)$ of the Jacobian matrix crosses zero in response to a change of parameter p . Without loss of generality, we normalize to zero the critical parameter value of bifurcation ($p = 0$) and we get generically two cases.

(1) Saddle-node bifurcation. A real eigenvalue crosses zero: $\lambda(0) = 0$.

(2) Hopf bifurcation. The real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero: $a(0) = 0$ and $b(p) \neq 0$ in a neighborhood of $p = 0$.

In discrete time, an elementary bifurcation occurs when one eigenvalue $\lambda(p)$ of the Jacobian matrix evaluated at the steady state, crosses the unit circle in response to a change of parameter p . Normalizing as above to zero the critical parameter value of bifurcation ($p = 0$), we find generically three classes of elementary bifurcations.

(1) Saddle-node bifurcation: $\lambda(0) = +1$.

(2) Flip bifurcation: $\lambda(0) = -1$.

(3) Hopf bifurcation: $|\lambda(0)| = |a(0) \pm ib(0)| = 1$ with $b(0) \neq 0$.

Generically, only one eigenvalue is concerned with a saddle-node or a flip bifurcation and the bifurcation analysis can reduce to the study of a simple one-dimensional invariant manifold. Similarly, two complex (conjugated) eigenvalues are concerned with the Hopf bifurcation and the bifurcation analysis simplifies to the study of a two-dimensional invariant manifold. More explicitly, when an eigenvalue (or a conjugated pair of eigenvalues in the case of Hopf) crosses the unit circle, generically, no other eigenvalue crosses simultaneously the circle. Then a higher-dimensional reduces to a one-dimensional dynamics or to a two-dimensional dynamics under a Hopf bifurcation (Central Manifold Theorem) and the movement of the other eigenvalues does not change the qualitative properties of dynamics (topological equivalence). In other terms, only a one or two-dimensional central manifold is concerned with the bifurcation: the other manifolds preserve their qualitative properties.

One-dimensional dynamics can not capture the occurrence of Hopf bifurcations. In order to study all the types of bifurcations in a common framework and highlight the differences between discrete and continuous time, we focus on two-dimensional systems. There is no loss of generality with respect to higher-dimensional dynamic system because of the Central Manifold Theorem.

3.3 Discretizations

We can compare discrete and continuous-time bifurcations through backward and forward-looking discretizations. While backward-looking approximations are more usual in other fields, such as physics, biology, demography, in economics expectations plays a role: agents face uncertainty and current decisions rest on the probabilities they have in mind about future. For instance, saving behavior depends on the expected interest rate. In order to capture forward-looking decision processes, forward-looking discretizations are more appropriate. Popular growth models mix backward and forward-looking mechanisms and are correctly approximated by hybrid discretizations, that is system of backward and forward-looking equations.

3.3.1 Backward-looking discretization

Focus first on (2). System $\dot{x} = f(x)$ and its approximation $x_{n+1} \approx x_n + hf(x_n)$ become, respectively:

$$\dot{x}_1 = f_1(x_1, x_2) \quad (16)$$

$$\dot{x}_2 = f_2(x_1, x_2) \quad (17)$$

and

$$x_{1n+1} \approx x_{1n} + hf_1(x_{1n}, x_{2n}) \equiv g_1(x_{1n}, x_{2n}) \quad (18)$$

$$x_{2n+1} \approx x_{2n} + hf_2(x_{1n}, x_{2n}) \equiv g_2(x_{1n}, x_{2n}) \quad (19)$$

As seen above, for both the systems, the steady state is the same and still given by

$$\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}$$

The Jacobian matrices of systems (16)-(17) and (18)-(19) are, respectively:

$$J_0 \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_{1n}} & \frac{\partial f_1}{\partial x_{2n}} \\ \frac{\partial f_2}{\partial x_{1n}} & \frac{\partial f_2}{\partial x_{2n}} \end{bmatrix} \text{ and } J_1 \equiv \begin{bmatrix} \frac{\partial g_1}{\partial x_{1n}} & \frac{\partial g_1}{\partial x_{2n}} \\ \frac{\partial g_2}{\partial x_{1n}} & \frac{\partial g_2}{\partial x_{2n}} \end{bmatrix} \quad (20)$$

Under a linear approximation, local dynamics in continuous and discrete time are related by h , the degree of discretization, according to a simple equality:

$$J_1 = I + hJ_0 \quad (21)$$

where I denotes a two-dimensional identity matrix.

We observe that, since J_0 depends on the steady state x which, in turn, does not depend on h , then J_1 depends only linearly on h as (21) shows.

In the following, let us denote the trace and determinant of J_0 and J_1 by (T_0, D_0) and (T_1, D_1) , respectively.

The characteristic polynomial in discrete time is given by $P_1(\lambda) \equiv \lambda^2 - T_1\lambda + D_1$, where

$$T_1 = 2 + hT_0 \quad (22)$$

$$D_1 = 1 + hT_0 + h^2D_0 = T_1 - 1 + h^2D_0 \quad (23)$$

In the case of a second-order discretization, we compute the Jacobian matrix of the RHS of (11) with respect to x_n and we find

$$J_1 = I + hJ_0 + \frac{h^2}{2} (J_0^2 + F_1H_{01} + F_2H_{02})$$

where J_0 is given in (20) and

$$\begin{aligned} F_1 &\equiv \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} & F_2 &\equiv \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix} \\ H_{01} &\equiv \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_{1n}^2} & \frac{\partial^2 f_1}{\partial x_{2n} \partial x_{1n}} \\ \frac{\partial^2 f_1}{\partial x_{1n} \partial x_{2n}} & \frac{\partial^2 f_1}{\partial x_{2n}^2} \end{bmatrix} & H_{02} &\equiv \begin{bmatrix} \frac{\partial^2 f_2}{\partial x_{1n}^2} & \frac{\partial^2 f_2}{\partial x_{2n} \partial x_{1n}} \\ \frac{\partial^2 f_2}{\partial x_{1n} \partial x_{2n}} & \frac{\partial^2 f_2}{\partial x_{2n}^2} \end{bmatrix} \end{aligned}$$

3.3.2 Hybrid discretization

In economics, many higher-dimensional models require a hybrid discretization to recover the equivalence between discrete and continuous time, that is a mix of discretization in backward and forward looking. Without loss of generality, we

consider a two-dimensional system where the first equation is discretized backward and the second one forward.⁵ Thus, the system of differential equations (16-17) becomes:

$$x_{1n+1} \approx x_{1n} + hf_1(x_{1n}, x_{2n}) \quad (24)$$

$$x_{2n+1} \approx x_{2n} + hf_2(x_{1n+1}, x_{2n+1}) \quad (25)$$

As seen above, the steady state is invariant to the choice of time and to the type of discretization (backward/forward): $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$.

The Jacobian matrix of the hybrid system (24)-(25) is:

$$J_1 = \begin{bmatrix} 1 + h \frac{\partial f_1}{\partial x_{1n}} & h \frac{\partial f_1}{\partial x_{2n}} \\ h \frac{\partial f_2}{\partial x_{1n+1}} \frac{1 + h \frac{\partial f_1}{\partial x_{1n}}}{1 - h \frac{\partial f_2}{\partial x_{2n+1}}} & \frac{1 + h^2 \frac{\partial f_2}{\partial x_{1n+1}} \frac{\partial f_1}{\partial x_{2n}}}{1 - h \frac{\partial f_2}{\partial x_{2n+1}}} \end{bmatrix} \quad (26)$$

The characteristic polynomial in discrete time is given by $P_1(\lambda) \equiv \lambda^2 - T_1\lambda + D_1$, where

$$T_1 = 2 + \frac{h}{1 - h \frac{\partial f_2}{\partial x_2}} (T_0 - hD_0) \quad (27)$$

$$D_1 = 1 + \frac{h}{1 - h \frac{\partial f_2}{\partial x_2}} T_0 \quad (28)$$

are the trace and the determinant.⁶

In the following, we study how conditions for elementary bifurcations change under a discretization of a continuous-time system. For simplicity, we focus on two-dimensional backward-looking discretizations, but results can be easily extended to the case of hybrid and higher-dimensional dynamic systems.

3.4 On the saddle-node equivalence

The continuous-time properties of the family of saddle-node bifurcations (saddle-node, transcritical and pitchfork) are preserved in discrete time. In a way, the saddle-node is the less sophisticated among the elementary bifurcations. The following proposition shows that this bifurcation persists under an Euler discretization.

Proposition 3 *A saddle-node bifurcation generically occurs in continuous time if and only if it arises in discrete time, whatever the discretization step h .*

⁵The reader is referred to the section on the Ramsey-Cass-Koopmans models.

⁶In the particular case $\partial f_2/\partial x_2 = 0$, we get

$$T_1 = 1 + D_1 - h^2 D_0 \quad (29)$$

$$D_1 = 1 + hT_0 \quad (30)$$

Proof. In continuous time, a saddle-node bifurcation occurs if both the eigenvalues are real and one generically crosses zero. In the bifurcation point, the determinant of the Jacobian matrix of system (16)-(17) becomes zero (if one eigenvalue is zero, also the product of eigenvalues is zero): $D_0 = 0$.

Consider now the discretization (18)-(19) with trace and determinant (22)-(23). It is known that in discrete time a saddle-node (saddle-node, transcritical or pitchfork) bifurcation generically arises if and only if $D_1 = T_1 - 1$ (simply notice that the characteristic polynomial evaluated at $\lambda = 1$ is zero: $P_1(1) = 1 - T_1 + D_1 = 0$).

Equality (23) implies that $D_0 = 0$ if and only if $D_1 = T_1 - 1$, whatever the discretization step, that is the proposition. ■

First, we notice that discretization preserves a saddle-node bifurcation, even if the approximation is extremely rough (whatever $h > 0$).

Second, we observe that, even if $D_0 = 0$ entails that $P_1(1) = 0$, that is the pair (T_1, D_1) lies on the line $D_1 = T_1 - 1$ in the (T, D) -plane (see the line AC in Figure 1), nevertheless the location on the line still depends on the discretization step, since $(T_1, D_1) = (2 + hT_0, 1 + hT_0)$, according to (22)-(23). In other terms, the equivalence stated in Proposition 3, refers only to the occurrence of a saddle-node, but does not rule out the role of h , the discretization step, in dynamics: if $\lambda_1 \equiv 1$, then $\lambda_2 = 1 + hT_0$ still depends on h .⁷

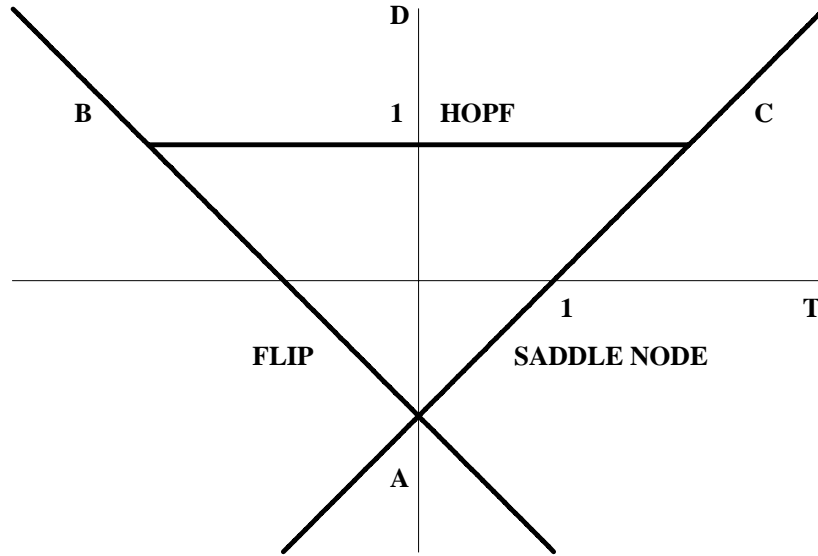


Figure 1. Bifurcation lines.

3.5 On the Hopf equivalence

Conditions for Hopf bifurcation in discrete time are "close" and tend to those in continuous time as the "distance" h between dynamics in continuous and

⁷The representation of bifurcations in the (T, D) -plane follows Samuelson (1941).

discretized time tends to zero.

Let us consider a parametrized function $f(x_1, x_2, p)$ with the following notation for derivatives:

$$\begin{aligned} f_{ix_j} &\equiv \partial f_i / \partial x_j \\ f_{ip} &\equiv \partial f_i / \partial p \end{aligned}$$

Proposition 4 Assume $f(x, p) \in C^2$. A Hopf bifurcation in continuous time generically requires $T_0 = 0$ and $D_0 > 0$; while, under a backward-looking discretization, it needs

$$T_0 = -hD_0 \quad (31)$$

$$D_0 \geq T_0^2/4 \quad (32)$$

where h is the discretization step. The right-hand sides of (31) and (32) tend to zero as h goes to zero, that is conditions (31) and (32) "converge" to the corresponding conditions in continuous time.

Proof. Given the Jacobian matrix, we compute the continuous-time characteristic polynomial: $P_0(\lambda) = \lambda^2 - T_0\lambda + D_0$, where T_0 and D_0 are the trace and determinant of J_0 (see the Jacobian matrix (20)). The two roots of the characteristic polynomial are: $\lambda = T_0/2 \pm \sqrt{T_0^2/4 - D_0}$. Roots are complex if and only if $D_0 > T_0^2/4$. In this case, the eigenvalues become $\lambda = \alpha \pm i\beta$ with $\alpha \equiv T_0/2$ and $\beta \equiv \sqrt{D_0 - T_0^2/4}$.

Hopf bifurcation in continuous time generically requires: $\alpha = 0$ and $\beta \neq 0$, that is

$$T_0 = 0 \quad (33)$$

$$D_0 > T_0^2/4 = 0 \quad (34)$$

Consider now discretization (18)-(19) with trace and determinant (22)-(23). It is known that a Hopf bifurcation generically arises in discrete time if and only if $D_1 = 1$ and $D_1 \geq T_1^2/4$ (complex and conjugated eigenvalues have the same modulus and cross together the unit circle if their product (determinant) is one). In other terms, conditions to get a Hopf bifurcation become $T_1^2 \leq 4$ and $D_1 = 1$. Using (22)-(23), we get

$$T_1^2 = (2 + hT_0)^2 \leq 4 \quad (35)$$

$$D_1 = 1 + h(T_0 + hD_0) = 1 \quad (36)$$

(36) gives $T_0 + hD_0 = 0$ or, equivalently,

$$h = -T_0/D_0 \quad (37)$$

Substituting (37) in (35), we obtain $(2 - T_0^2/D_0)^2 \leq 4$ or, equivalently, $0 \leq T_0^2/D_0 \leq 4$. The left-hand inequality means $D_0 > 0$. Therefore the right-hand inequality becomes $D_0 \geq T_0^2/4$.

Summing up, the necessary and sufficient conditions for a Hopf bifurcation in discrete time are, generically: $T_0 + hD_0 = 0$ and $D_0 \geq T_0^2/4$.

More explicitly, taking into account that the derivatives appearing in J_0 and then in (T_0, D_0) depend directly and indirectly (through the steady state) on the parameter value p , we have

$$T_0(x(p), p) = -hD_0(x(p), p) \quad (38)$$

$$D_0(x(p), p) \geq T_0(x(p), p)^2/4 \quad (39)$$

where $x(p)$ is a stationary state corresponding to the parameter value p . We observe that (39) implies $D_0 \geq 0$ and, therefore, (38) entails $T_0 \leq 0$ if (without loss of generality) $h > 0$.

The Hopf bifurcation value p_H solves (38). Under the hypotheses of the Implicit Function Theorem, equation (38) locally defines an explicit function $p_H = p_H(h)$.

Compare now these conditions with those required in continuous time to obtain a Hopf bifurcation (33)-(34):

$$\begin{aligned} T_0(x(p), p) &= 0 \\ D_0(x(p), p) &> T_0(x(p), p)^2/4 \end{aligned}$$

Conditions (38) and (39) "converge" continuously to (33) and (34), respectively, as h (the "distance" between the systems in discrete and continuous time) goes to zero.

More precisely, if $f(x, p) \in C^2$, $x(p)$ is generically continuous because $f(x, p) \in C^1$ (apply the Implicit Function Theorem to $f_1(x_1, x_2, p) = 0$ and $f_2(x_1, x_2, p) = 0$) and $p_H(h)$ is generically continuous because $f(x, p) \in C^2$ (apply the Implicit Function Theorem to (38)), that is to⁸

$$\begin{aligned} 0 &= f_{1x_1}(x(p), p) + f_{2x_2}(x(p), p) \\ &\quad + h[f_{1x_1}(x(p), p)f_{2x_2}(x(p), p) - f_{1x_2}(x(p), p)f_{2x_1}(x(p), p)] \end{aligned}$$

⁸Notice that

$$\begin{bmatrix} x'_1(p) \\ x'_2(p) \end{bmatrix} = - \begin{bmatrix} f_{1x_1} & f_{1x_2} \\ f_{2x_1} & f_{2x_2} \end{bmatrix}^{-1} \begin{bmatrix} f_{1p} \\ f_{2p} \end{bmatrix}$$

with $f_{1x_1}f_{2x_2} - f_{1x_2}f_{2x_1} \neq 0$, and

$$p'_H(h) = - \frac{f_{1x_1}f_{2x_2} - f_{1x_2}f_{2x_1}}{\Delta}$$

with

$$\begin{aligned} \Delta &\equiv (1/h + f_{1x_1})(f_{2x_2x_1}x'_1 + f_{2x_2x_2}x'_2 + f_{2x_2p}) \\ &\quad + (1/h + f_{2x_2})(f_{1x_1x_1}x'_1 + f_{1x_1x_2}x'_2 + f_{1x_1p}) \\ &\quad - f_{1x_2}(f_{2x_1x_1}x'_1 + f_{2x_1x_2}x'_2 + f_{2x_1p}) \\ &\quad - f_{2x_1}(f_{1x_2x_1}x'_1 + f_{1x_2x_2}x'_2 + f_{1x_2p}) \\ &\neq 0 \end{aligned}$$

Continuity of x and p_H implies, respectively: $\lim_{p \rightarrow p_H} x(p) = x(p_H)$ and $\lim_{h \rightarrow 0^+} p_H(h) = p_H(0)$.

We obtain $\lim_{h \rightarrow 0^+} [hD_0(p_H(h), x(p_H(h)))] \rightarrow 0^-$ entailing that (38) "converges" to (33). (39) "converges" to (34) because when T_0 goes to zero, generically D_0 does not go to zero (the case where both the eigenvalues of J_0 are zero is non-generic). ■

In other terms, if a Hopf bifurcation arises in continuous time, it is (generically) possible to find a (sufficiently small) discretization step which preserves (by continuity) this bifurcation. Condition for Hopf in discrete time can be made arbitrarily close to that in continuous time: simply reduce the period length h (which is an inverse measure of the approximation degree of a continuous-time dynamics through a linear Euler discretization). Under nice continuity properties (namely, $f(x, p) \in C^2$), the discrete-time critical value $p_H(h)$ lies in a neighborhood of the continuous-time critical value $p_H(0)$.⁹

In some respect, there is no qualitative difference between the Hopf bifurcations in continuous and discrete time.

Finally, we notice that in the case of saddle-node bifurcations, the critical value p_S does not depend on h , while now h matters.

3.6 On the flip singularity

As seen above, the saddle-node bifurcation is invariant with respect to a linear Euler discretization.

The Hopf bifurcation is characterized by a qualitative equivalence in continuous or in discrete time and the continuous-time condition is generically obtained by continuity as the (linear) discretization parameter h converges to zero.

The very difference between dynamics in continuous and discrete time arises with the flip bifurcation under Euler approximations of the system in continuous time.

More precisely, we will prove that, under linear and higher-order Euler approximations, if the continuous-time eigenvalue is bounded from below, the flip bifurcation disappears in discrete time when the discretization parameter h falls below a positive lower bound h^* .

We know also that the (vector) variation $x_{n+1} - x_n = x(nh + h) - x(nh)$ in continuous time is exactly represented by an infinite order Taylor series, provided that the original function f belongs to the class C^∞ . A plausible conjecture is that the critical discretization step h^* depends on the order q of Taylor discretization (see expression (15): $h^* = h^*(q)$), increases with respect to q and, eventually, $\lim_{q \rightarrow +\infty} h^*(q) = +\infty$ (when the approximation becomes infinitely accurate, there is no longer room for flip bifurcations whatever the

⁹In the particular case

$$\partial f / \partial x_2 = 0 \tag{40}$$

linearizing the hybrid discretization (24)-(25) gives $T_2 = 2 + hT_0 - h^2D_0$ and $D_2 = 1 + hT_0$. If $T_0 = 0$ and $D_0 > 0$ (conditions for Hopf bifurcation in continuous time, see (33)-(34)), we get also $D_1 = 1$ and $T_2 = 2 - h^2D_0 < 0$, satisfying conditions (37)-(35) for a Hopf bifurcation in discretized time whatever h . In other terms, under condition (40), we no longer requires the discretization step to be sufficiently small in order the Hopf equivalence to hold.

This case matters in growth theory because Ramsey-like models such as Cass-Koopmans (1965) or Zhang (2000) satisfy (40).

amplitude of h (even if the discretization step is arbitrarily large)). In order to prove this conjecture, one can focus on the one-dimensional case without loss of generality and apply the expansion (15): $x_{n+1} - x_n = \varphi(h) = \sum_{p=0}^q \varphi^{(p)}(0) h^p / p!$ (see also Krivine, Lesne and Treiner (2007) and Hale and Koçak (1991)).

In the following, we will highlight the role of h in the occurrence of flip bifurcation and we will confine ourselves to study, without great loss of generality (because of the Central Manifold Theorem), one-dimensional discretizations (first, second and higher-order expansions).

3.6.1 First-order discretization

A continuous-time scalar system: $\dot{x} = f(x, p)$, where p is the bifurcation parameter, can be approximated by a first-order Taylor polynomial: $x_{n+1} \approx x_n + hf(x_n, p) \equiv g(x_n, p)$.

One-dimensional dynamics Consider a parametrized steady state: $f(x, p) = 0$. In order to lighten notation, we will denote partial derivatives as follows: $f_x \equiv \partial f / \partial x$, $f_p \equiv \partial f / \partial p$, $f_{xx} \equiv \partial^2 f / \partial x^2$, $f_{pp} \equiv \partial^2 f / \partial p^2$ and so on.

As seen above, under the assumptions of the Implicit Function Theorem, the stationary state depends on the bifurcation parameter: $x = x(p)$.¹⁰

A flip bifurcation generically requires: $\lambda = g_x(x(p), p) = -1$ or, more explicitly:

$$\left. \frac{\partial [x_n + hf(x_n, p)]}{\partial x_n} \right|_{x_n=x(p)} = 1 + hf_x(x(p), p) = -1 \quad (41)$$

Applying the Implicit Function Theorem to (41), we get, locally, the critical value as a function of discretization degree: $p_F = p_F(h)$.¹¹

Sufficient conditions to rule out flip bifurcations In the following, without loss of generality, we set $h > 0$ and we call $X(p) \equiv \{(x, p) : f(x, p) = 0\}$ the set of stationary states x corresponding to a parameter value p . $X(P) \equiv \bigcup_{p \in P} X(p)$ is the graph of the stationary states obtained by varying the (scalar) parameter p . $X(p)$ is empty, when the system admits no stationary states at p . In the sequel, we consider only the range of parameter values generating a nonempty set of stationary states: $P \equiv \{p : X(p) \neq \emptyset\}$. The next proposition provides a sufficient condition to exclude flip bifurcations.

¹⁰If $f \in C^1$ and $f_x \neq 0$, we get $x'(p) = -f_p/f_x$.

¹¹If $f \in C^2$, $f_x \neq 0$ and $f_{xx}/f_x \neq f_{px}/f_p$, then

$$p'_F(h) = \frac{1}{h} \frac{f_x/f_p}{f_{xx}/f_x - f_{xp}/f_p} \quad (42)$$

Proposition 5 *Let $f \in C^1$ and $Y \equiv f_x(X(P))$. If $-\infty < \inf Y$, there exists a nonempty discretization range $(0, h^*)$ with $h^* \equiv |-2/\inf Y|$, where no flip bifurcation arises.*

Proof. (1) If $\inf Y \geq 0$, then $1 + hf_x(x(p), p) > 0 > -1$ whatever h and whatever the selection $(x(p), p) \in X(p)$. (2) If $\inf Y < 0$, solve $1 + h \inf Y > -1$ in order to exclude the flip bifurcation, that is, set $h < -2/\inf Y$. Then, if $h < |-2/\inf Y|$, (1) and (2) are satisfied. ■

Since computing the graph $X(P)$ and its image with respect to f_x can be difficult, we provide another sufficient condition, less general than Proposition 5, but easier to check.

Corollary 6 *Let $f \in C^1$ and $Z \equiv f_x(S \times P)$, where S is the domain of x . If $-\infty < \inf Z$, then there exists a discretization range $(0, h^*)$ with $h^* \equiv |-2/\inf Z|$ with no flip bifurcation.*

Proof. Simply notice that $X(P) \subseteq S \times P$ and apply Proposition 5. ■

In order to prove the usefulness of Proposition 5 and its corollary, we provide two examples: the first one shows how the boundedness of parameter range P can account for the lack of flip bifurcations in a nonempty interval $(0, h^*)$; the second example shows the same for an unbounded parameter range.

Example with bounded parameter range Consider a quadratic dynamics in continuous time: $\dot{x} = f(x, p) = px - x^2$, with two steady states: $x_0 = 0$ and $x_1 = p$. Euler discretization gives: $x_{n+1} \approx x_n + hf(x_n, p) = x_n + h(px_n - x_n^2) \equiv g(x_n, p)$. Flip bifurcation requires (41), that is

$$1 + h[p - 2x(p)] = -1 \quad (43)$$

We apply this condition to the two steady states.

(1) In the steady state $x_0 = 0$, (43) becomes $1 + hp = -1$, that is $p_F = -2/h$. So, there is a flip bifurcation whatever $h > 0$, if $p \in (-\infty, b)$ with $0 \leq b \leq +\infty$. We observe that the continuous-time eigenvalue $f_x(0, p) = p$ is unbounded from below if $p \in (-\infty, b)$.

(2) In the other steady state, $x_1 = p$, (43) becomes $1 - hp = -1$, that is $p_F = 2/h$. Again, there is always a flip bifurcation whatever $h > 0$, if $p \in (a, +\infty)$ with $-\infty \leq a \leq 0$. The eigenvalue $f_x(p, p) = -p$ is unbounded from below if $p \in (a, +\infty)$.

Reconsider now both the cases ((1) and (2)) and place a restriction on the range of parameter values in order to obtain $h^* > 0$. Assume, for instance, $p \in [a, b]$ with $a, b \in R$. Then $f_x(0, p) = p \in [a, b]$ and $f_x(p, p) = -p \in [-b, -a]$, that is $Y \equiv f_x(X(P)) = [-b, -a] \cup [a, b]$ and Proposition 5 eventually applies: $h^* \equiv |-2/\inf Y|$.

A numerical example can clarify more this point. Let $a = 0, b = 1$. Then $Y = [-1, 1]$ and $h^* \equiv |-2/\inf Y| = 2$. Consider, for example, a linear discretization with $h = 1 \in (0, 2)$: $x_{n+1} \approx (1 + p)x_n - x_n^2 \equiv g(x_n, p)$. The flip condition (43) at $x_0 = 0$ becomes $1 + p = -1$ and we notice that $p = -2 \notin [a, b] = [0, 1]$.

Similarly, the flip condition (43) at $x_1 = p$ becomes $1 - p = -1$ and we observe that $p = 2 \notin [a, b] = [0, 1]$. Summing up, no flip bifurcation arises at $h = 1$ whatever the steady state we consider.

Example with unbounded parameter range In economic applications, some parameters are unbounded (e.g. the elasticities of substitution). If $\partial f/\partial x$ is bounded from below, Proposition 5 still applies.

In order to find an explicit solution, we consider a simple example: $\dot{x} = f(x, p) = 2(2p - x - 1/x)$ with $p \in P \equiv (-\infty, -1) \cup (1, +\infty)$. As above, we obtain two steady states: $x_1 \equiv p - \sqrt{p^2 - 1}$ and $x_2 \equiv p + \sqrt{p^2 - 1}$. Euler discretization becomes: $x_{n+1} \approx x_n + hf(x_n, p) = x_n + 2h(2p - x_n - 1/x_n) \equiv g(x_n, p)$. Flip bifurcation requires (41), that is

$$1 + 2h \left[x(p)^{-2} - 1 \right] = -1 \quad (44)$$

We apply this condition to both the steady states. The continuous-time eigenvalue f_x is bounded whatever $p \in P$:

$$f_x(x(p), p) = 2 \left[\left(p \pm \sqrt{p^2 - 1} \right)^{-2} - 1 \right] \in [-2, +\infty]$$

where the sign $- (+)$ refers to the steady state $x_1 (x_2)$. Then $Y \equiv f_x(X(P)) = [-2, +\infty]$ whatever $p \in P$, and $h^* \equiv |-2/\inf Y| = 1$.

A numerical example will convince the reader. Set, for instance, $h = 1/2 \in (0, 1)$. There is no room for flip bifurcations under a linear discretization: indeed, the flip condition (44) requires $\left(p \pm \sqrt{p^2 - 1} \right)^2 = -1$, where $- (+)$ still refers to the steady state $x_1 (x_2)$, that is an impossible equality, whatever $p \in P$.

Two-dimensional dynamics A flip bifurcation in discrete time requires $D_1 = -T_1 - 1$ or, more explicitly, according to (22-23):

$$D_0(p)h^2 + 2T_0(p)h + 4 = 0 \quad (45)$$

where $T_0(p)$ and $D_0(p)$ denote, with some notational misuse, the trace and determinant of the Jacobian matrix $J_0(p)$ in continuous time, which depends on the parameter p .

Thus, the bifurcation value in discrete time depends on h : $p_F = p_F(h)$. Nothing ensures that $D_0(p_F(h))h^2 + 2T_0(p_F(h))h$ goes to zero as h goes to zero. However, if the derivatives of the Jacobian matrix are bounded on $X(P)$, $\lim_{h \rightarrow 0^+} [D_0(p_F(h))h^2 + 2T_0(p_F(h))h + 4] = 4 > 0$ and, therefore, there exists a discretization degree h^* such that $h \in (0, h^*)$ rules out the occurrence of flip bifurcations.

The reader can also apply the arguments developed in the previous section, up minor changes, and derive sufficient conditions to show that cycles disappears when the discretization step reduces.

3.6.2 Second-order discretization

For simplicity, we focus only on one-dimensional dynamics. We provide sufficient conditions to rule out flip bifurcations under small discretization step, whatever the parameter value.

Proposition 7 *Let $f \in C^3$ on $S \times P$ and f, f_x, f_{xx} be bounded over $X(P)$. Then there exists a nonempty discretization range $(0, h^*)$, where generically no flip bifurcation arises.*

Proof. The continuous-time system $\dot{x} = f(x, p)$ can be approximated by a second-order Taylor polynomial (expansion (4)):

$$x_{n+1} \approx x_n + f(x_n, p)h + f(x_n, p)f_x(x_n, p)h^2/2 \equiv g(x_n, p) \quad (46)$$

The steady state solves $f(x, p) = 0$ and the solution is a correspondence $X(p)$ which depends on the parameter p . For simplicity, focus on a selection $x(p) \in X(p)$.¹²

A flip bifurcation generically arises at: $\lambda = g_x(x(p), p) = -1$. Using (46), more explicitly, we get

$$\begin{aligned} & g_x(x(p), p) \\ \equiv & \partial [x_n + f(x_n, p)h + f(x_n, p)f_x(x_n, p)h^2/2] / \partial x_n|_{x_n=x(p)} \\ = & 1 + f_x(x(p), p)h + \left([f_x(x(p), p)]^2 + f(x(p), p)f_{xx}(x(p), p) \right) h^2/2 \\ = & -1 \end{aligned} \quad (47)$$

In order to solve equation (47) and to find the critical value as a local function of the discretization degree: $p_F = p_F(h)$, we apply the Implicit Function Theorem to (47).¹³

If f, f_x, f_{xx} are bounded on the graph $X(P)$, the derivative appearing on the left-hand side in (47) goes to one as h goes to zero: $\lim_{h \rightarrow 0^+} g_x(x(p_F(h)), p_F(h)) = 1 > -1$ violating the flip condition on the right-hand side.

By continuity, there exists a (right) neighborhood of 0, $(0, h^*)$, where (47) is violated, whatever $p \in P$. ■

Proposition 8 *If P is compact and, for every $(x, p) \in S \times P$, $f \in C^2$ and f_x is nonzero, then no flip bifurcation occurs in a nonempty interval $(0, h^*)$.*

¹²As above, if $f \in C^1 (\supseteq C^3)$ and $f_x \neq 0$, we get $x'(p) = -f_p/f_x$.

¹³ $f(x, p) \in C^3$ is a comfortable assumption. Totally differentiating (47), we obtain

$$p'_F(h) = \frac{1}{h} \frac{f_x}{f_p} \frac{f_x + \frac{1}{h} + f \frac{f_{xx}}{f_x}}{\left(\frac{f_{xx}}{f_x} - \frac{f_{xp}}{f_p} \right) + \frac{f}{2} \left(\frac{f_{xxx}}{f_x} - \frac{f_{xpp}}{f_p} \right)}$$

All the derivatives are evaluated in $(x(p), p)$. We require the denominator of $p'_F(h)$ to be nonzero (generically).

Compare with (42).

Proof. $f \in C^2$ implies the continuity of f, f_x, f_p, f_{xx} . $f_x, f_p \in C^0$ and $f_x \neq 0$ over $S \times P$ entail the continuity of a selection $x(p)$ over P . Continuity of $f, f_x, f_{xx}, x(p)$ and compactness of P imply boundedness of $f(x(P), P), f_x(x(P), P), f_{xx}(x(P), P)$. Then

$$\lim_{h \rightarrow 0^+} \left[1 + f_x(x(p), p)h + \left([f_x(x(p), p)]^2 + f(x(p), p)f_{xx}(x(p), p) \right) h^2/2 \right] = 1$$

whatever $p \in P$ and (47) is violated in a (right) neighborhood of zero. ■

We observe that this procedure can be generalized to third-order discretization through the expansion

$$\begin{aligned} x_{n+1} &\approx x_n + f(x_n, p)h + f(x_n, p)f_x(x_n, p)h^2/2 \\ &\quad + \left[f(x_n, p)f_x(x_n, p)^2 + f(x_n, p)^2 f_{xx}(x_n, p) \right] h^3/6 \\ &\equiv g(x_n, p) \end{aligned}$$

of a continuous-time dynamics $\dot{x} = f(x)$. Without entering details, we notice that the restriction $f \in C^4$ is a comfortable assumption in order to apply, as above, the Implicit Function Theorem.

In the following, we will consider many popular dynamic models: in particular, the most influential dynamic models (Kaldor (1940), Solow (1956) and Cass-Koopmans (1965)), and we compare discrete-time and discretized dynamics with or without imperfections.

Part II

Economic applications

Dynamic complexity can emerge in simple economic models. Introducing market imperfections in general equilibrium models can promote the occurrence of different bifurcations.

Ramsey-type models without market imperfections are characterized by saddle-path stability. Flip bifurcations arise in Solow models with negative externalities (Day (1982)). Hopf bifurcations occur in Ramsey models with positive externalities (Zhang (2000)).

Discounting matters. Discretizing the continuous-time Cass-Koopmans model, we recover the discrete-time version, only if the discount rate approximation is appropriately made.

In order to bridge the growth literature, we provide a general model of intertemporal optimization and we present Zhang (2000) as a particular case. In addition, we recover the Cass-Koopmans model as a particular case of Zhang (2000) (when the externalities become zero) and we show that Ramsey (1928) is in turn a special case of Cass-Koopmans (1965) (when the discount rate becomes zero).

Among different questions, one issue is especially addressed and concerns the very nature of dynamics in models of growth. Indeed, beyond the variety of

models, capital accumulation is a common feature: the capital stock is a state variable either in Solow or in Ramsey models and its predetermined nature is properly represented through a backward-looking discretization. Only through such approximation the discrete-time capital accumulation can be recovered from the continuous-time law of motion.

However, in the class of Ramsey models, the intertemporal arbitrage replaces the assumption made by Solow of an exogenous saving rate. The very nature of utility maximization, which results in the intertemporal arbitrage, is forward, because the consumption the agent smooths, is a jump variable: in order to recover the discrete-time from the continuous-time Euler equation, a forward-looking discretization is needed.

Putting together capital accumulation and consumption smoothing, we obtain the two-dimensional dynamics of Ramsey models: the twofold nature of the system (backward and forward) is properly represented by a hybrid discretization.

4 Backward-looking discretizations

Discrete-time version of popular dynamic models such as Kaldor (1940) and Solow (1956) can be derived through a backward-looking (Euler) discretization. In order to show how the occurrence of cycles of period two (flip bifurcation) depends on the length of the discretization step, we introduce negative externalities in the seminal Solow model (Day (1982)). While Solow and Day dynamics are one-dimensional, the Kaldor model is two-dimensional. The main asset of Kaldor (1940) is the occurrence of Hopf bifurcations under a parsimonious set of assumptions.

4.1 Solow models

In the case of Solow models or Solow models with externalities, dynamics are one-dimensional and there is no room for Hopf bifurcations.

We compare the continuous and discrete-time versions of the most influential growth model and focus only on the occurrence of saddle-node and flip bifurcations.

In growth theory, utility maximization results in the household's consumption smoothing, that is a forward-looking behavior which is taken in account by a forward-looking discretization. In the Solow model, preferences are roughly summarized by an exogenous saving rate and there is no longer room for forward-looking mechanisms.

This explains why we recover the discrete-time Solow model through a (first-order) backward-looking discretization. However, we notice that discretizing dynamics in the capital intensity is not equivalent to discretizing dynamics in the aggregate capital.

4.1.1 Solow models

In continuous time, the seminal version is a two-dimensional dynamics:

$$\dot{K}_t = sF(K_t, L_t) - \delta K_t \quad (48)$$

$$\dot{L}_t = n_0 L_t \quad (49)$$

and reduces to an intensive law of motion under the assumption of a CRS technology:

$$\dot{k}_t = sf(k_t) - (\delta + n_0)k_t \quad (50)$$

where $k \equiv K/L$ is the capital intensity and $f(k) \equiv F(k, 1)$. s , δ and n_0 denote the saving rate, the rate of capital depreciation and the demographic growth rate, respectively. Capital letters stand for aggregate variables. Under the usual neoclassical assumptions on technology, the non-trivial steady state is unique and solves

$$f(k)/k = (\delta + n_0)/s \quad (51)$$

Moreover, it is locally stable because the eigenvalue of the intensive dynamics, evaluated at the steady state, is $\lambda_0 = -(1-a)(\delta + n_0) < 0$, where $a \equiv kf'(k)/f(k) \in (0, 1)$ is the capital share. In other terms, there is no room for local bifurcations.

In discrete time, the basic model is still two-dimensional:

$$K_{t+1} - K_t = sF(K_t, L_t) - \delta K_t \quad (52)$$

$$L_{t+1} - L_t = n_1 L_t \quad (53)$$

and reduces to the one-dimensional intensive law:

$$k_{t+1} = [sf(k_t) + (1 - \delta)k_t] / (1 + n_1) \quad (54)$$

where n_1 denotes the demographic growth rate. As above, the positive steady state solves $f(k)/k = (\delta + n_1)/s$ and is unique under the usual assumptions. Local stability is entailed by the eigenvalue in the unit circle:

$$\lambda_1 = 1 - (1 - a)(\delta + n_1) / (1 + n_1) \in (0, 1) \quad (55)$$

As above, there is no room for local bifurcations.

4.1.2 Discretized Solow models

The question we address now is whether Euler approximations (of different orders) are appropriate to recover the discrete-time model.

We begin with a first-order discretization of the original continuous-time system. Discretizing system (48)-(49) and referring to (18)-(19) with $x_1 \equiv K$ and $x_2 \equiv L$ gives

$$K_{n+1} \approx K_n + hf_1(K_n, L_n) = K_n + h[sF(K_n, L_n) - \delta K_n] \quad (56)$$

$$L_{n+1} \approx L_n + hf_2(K_n, L_n) = (1 + hn_0)L_n \quad (57)$$

Normalizing (56) by L_n , we derive the intensive law:

$$k_{n+1} \approx [(1 - h\delta)k_n + hsf(k_n)] / (1 + hn_0)$$

The steady state does not depend on the discretization step and still solves (51), while the eigenvalue depends on h (indeed the Euler method approximates the transition):

$$\lambda_1 = 1 - (1 - a)(\delta + n_0) / (1/h + n_0) \quad (58)$$

Under the assumption $h > 0$, $\lambda_1 < 1$ and there is no room for a saddle-node bifurcation.

If $n_0 < \delta(1 - a) / (1 + a)$, a flip bifurcation arises at

$$h_F = 2 / [(1 - a)\delta - (1 + a)n_0] > 2$$

The case $h = 1$ deserves a comment. In this case, there is no longer room for bifurcations and the discretized system (56)-(57), with $n = t$, reduces to (52)-(53), provided that $n_0 = n_1$. Thus, we can say that the traditional discrete-time Solow model is exactly the first-order discretization of the original continuous-time model.

We observe that, in general, computing the intensive form of discretized dynamics is not equivalent to discretizing the intensive form of the original model. Indeed, discretizing the intensive form (50) gives

$$k_{n+1} \approx k_n + h[sf(k_n) - (\delta + n_0)k_n]$$

The steady state remains the same (see (51)), but the eigenvalue $\lambda_1 = 1 - (1 - a)(\delta + n_0)h$ is now slightly different from those of the discrete-time model (55) and of the discretized model (58). As above, there is no room for a saddle-node bifurcation. However, the flip bifurcation is possible if the discretization step is sufficiently large: $h_F = 2 / [(1 - a)(\delta + n_0)]$. When $h = 1$, the law of motion reduces to $k_{t+1} \approx sf(k_t) + (1 - \delta - n_0)k_t$, which is different from that of the discrete-time model (54). A flip bifurcation is still possible at $n_0 = 2 / (1 - a) - \delta$.

Let us focus now on the second-order Euler discretizations.

Increasing the order of discretization makes the approximation closer to the continuous-time system. Intuitively, we conjecture that, when a discrete time system approaches continuous-time dynamics, it inherits their features. In particular, we expect that flip bifurcations disappear when the order of approximation is sufficiently high. Surprisingly, the next proposition proves that the order two is sufficient to rule out any flip bifurcations and two-period cycles.

Proposition 9 *In the Solow model with a CRS technology, the second-order Euler discretization of the original continuous-time system (48)-(49) reduces to the intensive law of motion:*

$$\begin{aligned} & k_{n+1} \\ \approx & \frac{k_n + [sf(k_n) - \delta k_n] (h + [sf'(k_n) - \delta] h^2 / 2) + sn_0 [f(k_n) - k_n f'(k_n)] h^2 / 2}{1 + hn_0 + (hn_0)^2 / 2} \end{aligned} \quad (59)$$

As above, the steady state solves (51), which no longer depends on h , while the eigenvalue is given by

$$\lambda_2 = \frac{1}{2} \frac{1 + (1 + h[an_0 - (1 - a)\delta])^2}{1 + hn_0 + (hn_0)^2/2} > 0 \quad (60)$$

The quadratic approximation (59) rules out any flip bifurcation and, for $h = 1$, even the saddle-node bifurcation is excluded.

Proof. In order to approximate the original continuous-time system (48)-(49), we apply formula (10) with $x_1 \equiv K$, $x_2 \equiv L$, $f_1(K_t, L_t) = sF(K_t, L_t) - \delta K_t$, $f_2(K_t, L_t) = n_0 L_t$:

$$\begin{bmatrix} K_{n+1} - K_n \\ L_{n+1} - L_n \end{bmatrix} \approx \begin{bmatrix} h + [s(\partial F/\partial K_n) - \delta]h^2/2 & s(\partial F/\partial L_n)h^2/2 \\ 0 & h + n_0 h^2/2 \end{bmatrix} \begin{bmatrix} sF(K_n, L_n) - \delta K_n \\ n_0 L_n \end{bmatrix} \quad (61)$$

Dividing the first equation by L_n , replacing the second equation $L_{n+1}/L_n = 1 + hn_0 + (hn_0)^2/2$ in the first and noticing that, under the assumption of CRS, $\partial F/\partial K_n = f'(k_n)$ and $\partial F/\partial L_n = f(k_n) - k_n f'(k_n)$, we obtain the intensive discretization (59).

Equation (59) becomes at the steady state:

$$1 \approx \frac{1 + h[sf(k)/k - \delta] + [sf(k)/k - \delta][asf(k)/k - \delta] + (1 - a)n_0 sf(k)/k}{1 + hn_0 + (hn_0)^2/2} \quad (62)$$

where $a \equiv kf'(k)/f(k)$. A way of checking that the steady state of the continuous-time model remains the steady state of our quadratic approximation is substituting (51) in (62): after tedious computations, the RHS simplifies to one.

Deriving the RHS of (59) w.r.t. k_n and evaluating the derivative at the steady state, that is replacing (51) and $a \equiv kf'(k)/f(k)$, after further tedious computations, we obtain (60). $\lambda_2 > 0$ prevents the occurrence of flip bifurcations. From (60), simple computations show that $h = 1$ implies $\lambda_2 \in (0, 1)$, that is a monotonic convergence to the steady state. ■

As above, computing the quadratic approximation of the original continuous-time system and, then, deriving the intensive form, is different from computing the intensive form and, then, its quadratic approximation.

Reconsider the continuous-time intensive form (50) and apply the expansion (4).

$$k_{n+1} \approx k_n + [sf(k_n) - (\delta + n_0)k_n] \left(h + \frac{h^2}{2} [sf'(k_n) - (\delta + n_0)] \right) \quad (63)$$

The steady state remains the same as in the continuous-time model and no longer depends on h (see (51)). Deriving the RHS of (63) and using the information of steady state (51), we get the eigenvalue:

$$\lambda_2 = \left(1 + [1 - h(1 - a)(\delta + n_0)]^2\right) / 2 > 0$$

where a is the capital share. Notice that a saddle-node bifurcation can arise at $h_S = 2 / [(1 - a)(\delta + n_0)]$, but, since $\lambda_2 > -1$, whatever $h > 0$, there is no longer room for cycles.

4.2 Solow models with externalities

Introducing imperfections in the Solow model can promote non-monotonic dynamic. Considering externalities is a simple way of generating complex dynamics and, possibly, chaos.

Let us focus on negative productive externalities from a firm to another and assume that the environmental quality enhances factors' productivity, but is, in turn, negatively affected by the average capital intensity. Formally, capital intensity k reduces the environmental quality to $m - k^{1-a}$, where $m > 0$ is the endowment of quality.

For simplicity, we assume a Cobb-Douglas production function (as in Day (1982)) and introduce an upper bound for the negative externality to ensure a positive TFP. More precisely:

Assumption 1 *The production function is given by*

$$F(K_t, L_t) \equiv A (m - k_t^{1-a}) K_t^a L_t^{1-a} \quad (64)$$

with $k_0 \in [0, m^{1/(1-a)}]$.

4.2.1 Day models

In this section, we highlight the differences between continuous and discrete time: introducing externalities matters and the dynamic equivalence between the traditional Solows models no longer holds.

In continuous time, the dynamics system becomes

$$\dot{K}_t = sA \left[m - (K_t/L_t)^{1-a} \right] K_t^a L_t^{1-a} - \delta K_t \quad (65)$$

$$\dot{L}_t = n_0 L_t \quad (66)$$

because of the equilibrium condition $sF(K_t, L_t) = \dot{K}_t + \delta K_t$ (saving = investment).

The dynamics of capital intensity reduce to

$$\dot{k}_t \equiv sA (m - k_t^{1-a}) k_t^a - (\delta + n_0) k_t \quad (67)$$

with a (non-trivial) steady state:

$$k = [msA / (\delta + n_0 + sA)]^{1/(1-a)} \in [0, m^{1/(1-a)}] \quad (68)$$

The explicit solution of (67) is given by:

$$k_t = \left[k^{1-a} + (k_0^{1-a} - k^{1-a}) e^{-(1-a)(\delta+n_0+sA)t} \right]^{1/(1-a)}$$

Under Assumption 1, the entire sequence of capital intensities lies in the interval $[0, m^{1/(1-a)}]$ and the steady state k is asymptotically stable, that is the capital intensity converges monotonically towards its stationary value in the long run: $\lim_{t \rightarrow +\infty} k_t = k$.

Therefore, in continuous time there is no room for bifurcations.

Conversely, in discrete time, persistent cycles and, possibly, chaos can arise. In order to prove this crucial difference, reconsider the Solow model (52)-(53) augmented with productive externalities (64). Under the Assumption 1, normalizing (52) by L_t and replacing L_{t+1}/L_t with $1 + n_1$, gives the law of motion of capital intensity:

$$k_{t+1} = [(1 - \delta - sA) k_t + sAmk_t^a] / (1 + n_1) \quad (69)$$

with steady state

$$k = [msA / (\delta + n_1 + sA)]^{1/(1-a)} \in [0, m^{1/(1-a)}] \quad (70)$$

We observe that (68) coincides with (70) iff $n_0 = n_1$.

The eigenvalue evaluated at the steady state is less than one: $\lambda_1 = a + (1 - a)(1 - \delta - sA) / (1 + n_1) < 1$, and, thus, there is no room for a saddle node. However, a flip bifurcation ($\lambda_1 = -1$) generically occurs at

$$A = A_F \equiv \frac{1}{s} \left[1 - \delta + (1 + n_1) \frac{1 + a}{1 - a} \right] \quad (71)$$

In other terms, negative productive externalities generate cycles (when production increases, capital intensity goes up, productivity is lowered by the externality and, eventually, production as well).

4.2.2 Discretized Day models

One may question whether discretizing the original continuous-time system (65)-(66) introduces new bifurcations in the model. Let us consider a first-order discretization. System (56)-(57) still holds. h denotes the discretization step, that is $x_n \equiv x(t_n) = x(nh)$ with $x = K, L$. Normalizing equation (56) by L_n , using (57) to find $L_{n+1}/L_n = 1 + hn_0$, and replacing in (56), under the Assumption 1, we obtain the intensive form: $k_{n+1} \approx [(1 - h\delta - hsA) k_n + hsAmk_n^a] / (1 + hn_0)$.

Setting $h = 1$ and $n_0 = n_1$, we recover exactly the discrete-time Day model (equation (69)). The steady state is still given by (68) or by (70) with $n_0 = n_1$ and does not depend on h .

As proved in the section "On the saddle-node equivalence", discretizing a system does not introduce saddle-node bifurcations. In our case, local dynamics are captured by the eigenvalue evaluated at the steady state: $\lambda_1 = 1 - h(1-a)(\delta + n_0 + sA)/(1 + hn_0) < 1$.

Conversely, a flip bifurcation ($\lambda_1 = -1$) generically arises at

$$A = A_F \equiv \frac{1}{s} \left(\frac{2}{1-a} \frac{1 + hn_0}{h} - \delta - n_0 \right) \quad (72)$$

Unsurprisingly, this critical value reduces to (71), when $h = 1$ and $n_0 = n_1$. Choosing the discretization step, we get equivalently

$$h_F = 2 / [(1-a)(\delta + n_0 + sA) - 2n_0]^{-1}$$

As in the Solow case, discretizing the original system and computing the intensive form is not equivalent to the reversed procedure, that is, discretizing the intensive dynamics.

The first-order discretization of (67) is given by

$$k_{n+1} \approx k_n + h [sA (m - k_n^{1-a}) k_n^a - (\delta + n_0) k_n] \equiv g(k_n)$$

while the steady state remains (68) (Euler discretization preserve the continuous-time steady state and the steady state is, obviously, independent of the discretization step h).

In order to ensure the positivity of variables under this linear Euler discretization and, in particular, $k_n \in [0, m^{1/(1-a)}]$ for $n = 0, 1 \dots$, we introduce a restriction in the parameter range.

Assumption 2

$$\frac{1 + a(1-a)^{1/a-1} (hsA)^{1/a}}{\delta + n_0 + sA} < h \leq \frac{1}{\delta + n_0} \quad (73)$$

Proposition 10 *If (73) holds, then $k_0 \in [0, m^{1/(1-a)}]$ implies $k_n = g^n(k_0) \in [0, m^{1/(1-a)}]$ for any integer n .*

Proof. We apply the induction principle. We need to prove that $k_n \in [0, m^{1/(1-a)}]$ entails $k_{n+1} \in [0, m^{1/(1-a)}]$.

First, we notice that $0 < k_n < m^{1/(1-a)}$ and $h \leq 1/(\delta + n_0)$ implies $k_{n+1} = g(k_n) > 0$.

Second, we observe that g is concave and that, since $h > \delta + n_0 + sA$, g attains its maximum at

$$k^* = \left[am \frac{hsA}{h(\delta + n_0 + sA) - 1} \right]^{1/(1-a)}$$

A sufficient condition for $k_{n+1} = g(k_n) < m^{1/(1-a)}$ is $g(k^*) < m^{1/(1-a)}$, or, equivalently, the left-hand inequality of (73). ■

Under condition (73), the interval $[0, m^{1/(1-a)}]$ is mapped into itself. The eigenvalue evaluated at the steady state is given by

$$\lambda_1 = 1 - h(1-a)(\delta + n_0 + sA) < 1$$

There is no room for a saddle-node bifurcation: $\lambda_1 < 1$, but there is for a flip bifurcation at $h_F = 2/[(1-a)(\delta + n_0 + sA)]^{-1}$ or, equivalently, at:

$$A = A_F = \frac{1}{s} \left[\frac{2}{h(1-a)} - \delta - n_0 \right] \quad (74)$$

h_F is required to satisfy (73). A_F is slightly different from (72).

One may question what dynamics arise if (73) is violated and, in particular, if $h < 1/(\delta + n_0 + sA)$. In this case, g becomes increasing and persistent cycles are ruled out. If $\delta + n_0 + sA < 1$, cycles disappear at $h = 1$.

For example, if $a = 1/2$, (73) becomes explicitly $h \leq 1/(\delta + n_0)$ and $A^- < A < A^+$, where $A^\pm \equiv 2 \left[1 \pm \sqrt{h(\delta + n_0)} \right] / (hs)$. These conditions are sufficient to ensure that $[0, m^2]$ is mapped in itself. However, the feasibility of the flip bifurcation requires an additional (sufficient) conditions $h_F \leq 1/(\delta + n_0)$ and $A^- < A_F < A^+$. The feasibility of the bifurcation point (74) requires $3(\delta + n_0)/s \leq A_F$ and $A^- < A_F < A^+$, where A_F is given by (74).

As in the case of Solow models, we can approximate better the continuous-time Day model with a second-order discretization. However, discretizing the original system is not equivalent to discretizing the intensive dynamics (67).

Proposition 11 *In the Day model with a Cobb-Douglas technology, the second-order Euler discretization of the original continuous-time system (65)-(66) reduces to the intensive law of motion:*

$$\begin{aligned} & k_{n+1} \\ = & \frac{k_n + [sAk_n^a m - (\delta + sA)k_n] [h + (asAk_n^{a-1}m - \delta - sA)h^2/2] + (1-a)n_0sAk_n^a m h^2/2}{1 + hn_0 + (hn_0)^2/2} \end{aligned} \quad (75)$$

The steady state is still given by (68) and no longer depends on h , while the eigenvalue is given by

$$\lambda_2 = \frac{1}{2} \frac{1 + (1 + h[an_0 - (1-a)(\delta + sA)])^2}{1 + hn_0 + (hn_0)^2/2} > 0 \quad (76)$$

The quadratic approximation (75) rules out any flip bifurcation.

Proof. In order to approximate the original continuous-time system (65)-(66), we apply formula (10) with $x_1 \equiv K$, $x_2 \equiv L$,

$$f_1(K_t, L_t) \equiv sA \left[m - (K_t/L_t)^{1-a} \right] K_t^a L_t^{1-a} - \delta K_t$$

$f_2(K_t, L_t) \equiv n_0 L_t$ and we obtain (61), where now F is given by (64).

Dividing the first equation in (64) by L_n , replacing the second equation $L_{n+1}/L_n = 1 + hn_0 + (hn_0)^2/2$ in the first and noticing that, under the assumption of private CRS, $\partial F/\partial K_n = aAk_n^{a-1}m - A$ and $\partial F/\partial L_n = (1-a)Ak_n^a m$, we get the intensive second-order discretized version of the augmented Solow model (75).

After tedious computations, we recover, as usual, the continuous-time steady state (68).

Eventually, we compute the eigenvalue. Deriving the RHS of (75) wrt k_n and evaluating it at the steady state, after further tedious computations, we obtain (76) which is formally close to (60). The main difference is due to the externalities sA . ■

As above, the quadratic approximation of the original continuous-time system is different from the quadratic approximation of intensive law.

Reconsider the continuous-time intensive form (67) and apply the expansion (4).

$$k_{n+1} \approx k_n + [msAk_n^a - (\delta + n_0 + sA)k_n] [h + (amsAk_n^{a-1} - \delta - n_0 - sA)h^2/2] \quad (77)$$

The steady state remains the same of the continuous-time model and no longer depends on h (see (68)). Deriving the RHS of (77) at the steady state, we get the eigenvalue: $\lambda_2 = \left(1 + [1 - h(1-a)(\delta + n_0 + sA)]^2\right)/2 > 0$. Notice that a saddle-node bifurcation can arise at $h_S = 2/[(1-a)(n_0 + \delta + sA)]$, but, since $\lambda_2 > -1$, there is no longer room for cycles.

Summing up, we can conclude that (1) in the Solow model, as well as in the Day model, it is not equivalent to discretize the original form or the intensive law; (2) the representation in discrete time is richer in terms of bifurcations.

In other terms, economic results are sensitive to the choice of time representation which is robust only if the conditions of equivalence between discrete and continuous time are satisfied. These conditions are quite specific.

4.3 Kaldor models

4.3.1 Kaldor models

We consider the continuous and discrete-time version of the popular Kaldor model (1940). In the spirit of Solow (1956), we do not require a microfoundation of the consumers' behavior which is simply summarized by an exogenous aggregate saving function $S(Y_t)$ with marginal propensity less than one: $0 < S_Y(Y_t) < 1$ for every $Y_t > 0$.

From our point of view, the main asset of the Kaldor model are two-dimensional dynamics where eigenvalues can be complex and Hopf bifurcations arise.

Kaldor pioneered the theory of business cycle from a Keynesian point of view. In the early Seventies Chang and Smyth (1971) provided a rigorous presentation

of the model:

$$\dot{K}_t = I(K_t, Y_t) - \delta K_t \quad (78)$$

$$\dot{Y}_t = p[I(K_t, Y_t) - S(Y_t)] \quad (79)$$

where $p > 0$, $I_K < 0$, $I_Y > 0$.

Equation (78) is the law of motion of physical capital, while equation (79) captures the response of output to an excess investment ($I_t - S_t$): when savings exceed the investment demand, the economy experiences a slowdown.

A steady state of system (78)-(79) is given by: $I(K, Y) = S(Y) = \delta K$. The trace and the determinant of the associated Jacobian matrix

$$J_0 \equiv \begin{bmatrix} I_K - \delta & I_Y \\ pI_K & p(I_Y - S_Y) \end{bmatrix} \quad (80)$$

are, respectively:

$$T_0 = I_K - \delta + p(I_Y - S_Y) \quad (81)$$

$$D_0 = p[(I_K - \delta)(I_Y - S_Y) - I_K I_Y] \quad (82)$$

Assumption 3 $0 < I_Y - S_Y < I_K I_Y / (I_K - \delta)$.

Inequality $I_Y - S_Y < I_K I_Y / (I_K - \delta)$ is equivalent to $D_0 > 0$. Under Assumption 3, the following remarks hold.

(0) The eigenvalues have the same sign and the steady state can never be a saddle.

(1) If $T_0 < 0$, both the eigenvalues have a negative real part and the steady state is stable (sink).

(2) If $T_0 > 0$, both the eigenvalues have a positive real part and the steady state is unstable (source).

(3) There is room for a Hopf bifurcation when T_0 crosses zero and the imaginary part is nonzero (the complex and conjugated eigenvalues cross the imaginary axis).

The parameter p is an appropriate bifurcation parameter: it can be interpreted as the "speed" of output response to excess savings.

In particular, one of the assets of Kaldor model is that limit cycles through a Hopf bifurcation easily occur. The critical value for a Hopf bifurcation is determined solving $T_0 = 0$:

$$p_{0H} = \frac{\delta - I_K}{I_Y - S_Y} \quad (83)$$

and is strictly positive under Assumption 3.

For $p < p_{0H}$, the real part of the roots is negative and the steady state is asymptotically stable, while it becomes positive for $p > p_{0H}$ (instability).

4.3.2 Discretized Kaldor models

One may question whether discretizing the continuous-time Kaldor model excludes the occurrence of Hopf bifurcation. Applying our general method, the following proposition holds.

Proposition 12 *The backward-looking discretized Kaldor model is*

$$K_{n+1} \approx K_n + h [I(K_n, Y_n) - \delta K_n] \quad (84)$$

$$Y_{n+1} \approx Y_n + hp [I(K_n, Y_n) - S(Y_n)] \quad (85)$$

The trace and the determinant evaluated at the steady state are $T_1 = 2 + hT_0$ and $D_1 = 1 + hT_0 + h^2D_0$ (equations (22) and (23)), where T_0 and D_0 are given by (81) and (82).

A Hopf bifurcation occurs at the critical value

$$p_{1H} = \frac{\delta - I_K}{I_Y - S_Y + h[(I_K - \delta)(I_Y - S_Y) - I_K I_Y]} \quad (86)$$

provided that

$$p_{1H} \geq p_{0H} - \frac{1}{h} \frac{4}{I_Y - S_Y}$$

p_{1H} is always positive for positive h 's under Assumption 3.

Proof. System (84)-(85) is simply obtained applying (18)-(19) to system (78)-(79). A Hopf bifurcation arises in discrete time if $D_1 = 1$ and $T_1^2 \leq 4$. The critical point p_{1H} is found replacing expressions (81)-(82) in equation $D_1 = 1$. The numerator of p_{1H} is positive; the denominator is positive under Assumption 3.

Noticing that $T_1 = 2 + hT_0$, condition $T_1^2 \leq 4$ is equivalent to $-4/h \leq T_0 \leq 0$. Replacing (81) and solving with respect to p , we get under Assumption 3:

$$\frac{\delta - I_K}{I_Y - S_Y} - \frac{1}{h} \frac{4}{I_Y - S_Y} \leq p_{1H} \leq \frac{\delta - I_K}{I_Y - S_Y}$$

that is

$$p_{0H} - \frac{1}{h} \frac{4}{I_Y - S_Y} \leq p_{1H} \leq p_{0H}$$

Assumption 3 ensures that $p_{1H} \leq p_{0H}$. This condition implies that the critical Hopf bifurcation value in discrete time is less than the corresponding value in continuous time, for any positive value of h . ■

These results deserve two comments.

(1) As seen in the theoretical part (Proposition 4), we find from (86) that p_{1H} converges exactly to p_{0H} when the discretization step tends to zero: $\lim_{h \rightarrow 0^+} p_{1H} = p_{0H}$.

(2) Setting $h = 1$ in system (84)-(85), we recover the traditional Kaldor model in discrete time. Thus, for $h = 1$, the discrete-time and the continuous-time critical values are different.

5 Hybrid discretizations

On the one hand, in Solow-like models, the saving rate is given and does not result from a utility maximization. There is no intertemporal optimization, there are no jump variables and the future is determined by the state variable. As seen above, the discrete-time model is recovered through a simple backward-looking discretization of the continuous-time model.

On the other hand, hybrid discretizations are important in economic theory when agents' behavior results from a dynamic optimization. Households smooth consumption over time under a budget constraint with the wealth inherited from the past (backward-looking side), while considering the future interest rate in their intertemporal arbitrage (forward-looking side). The twofold nature of the dynamic system becomes explicit when we discretize the continuous-time model. In order to recover the discrete-time model we need to discretize backward the budget constraint and forward the Euler equation (intertemporal smoothing). Influential examples of dynamic optimization are Ramsey (1928) and Cass-Koopmans (1965).¹⁴

5.1 Optimization models

Before entering (neo)classical growth models, we consider a theoretical approach to dynamic optimization. Solutions we provide are very general and can be applied to a large class of intertemporal optimization programs with intertemporal separability, beyond the economic applications. Indeed, we consider the possibility that both the state and the control variables enter either the objective functional or the constraint.

Optimal growth models work as particular cases and can be recovered as applications of a common theoretical core with no redundancies. So, in order to simplify the reader's task, we have decided to present the method, which remains merely theoretical, in the part devoted to economic applications.

In the next section, we solve the continuous and discrete-time programs, while, in the following, we discretize and linearize the system in continuous time.

5.1.1 Optimization models

We maximize a general intertemporal functional

$$V \equiv \int_0^{\infty} \beta_t v(k_t, c_t) dt \quad (87)$$

¹⁴ There are models with jump variable and without control (that is, without intertemporal optimization). Hybrid discretization still works because of the jump variable (forward-looking side) as, for instance, in Dornbusch (1976), which is a two-dimensional model where the commodity prices are predetermined, while the exchange rate jumps.

where k_t and c_t denote a state and the control, subject to a law of motion

$$\dot{k}_t \leq s(c_t, k_t) \quad (88)$$

and a discounting process $\dot{\beta}_t = -\rho_t \beta_t$, where $\rho_t = \rho(t)$ is a given positive function. The initial conditions k_0 and $\beta_0 \equiv 1$ are also given.

Assumption 4 *v and s are C^2 , strictly increasing in both the arguments ($\partial v/\partial k > 0, \partial v/\partial c > 0$) and strictly concave.¹⁵ The Inada boundary conditions are also satisfied.*

The agent chooses the control in order to maximize the functional subject to the law of motion.

β_t is a general discount function which depends on the lapse of time. Alternatively, we define the discount rate as

$$\rho_t \equiv -\dot{\beta}_t/\beta_t \quad (89)$$

We notice that, if $\rho_t = \rho$, a constant, then the solution of (89) is $\beta_t = \beta_0 e^{-\rho t}$. In this case, maximizing $\int_0^\infty e^{-\rho t} v(k_t, c_t) dt$ is equivalent to maximizing $\int_0^\infty \beta_t v(k_t, c_t) dt = \beta_0 \int_0^\infty e^{-\rho t} v(k_t, c_t) dt$.

The discounted Hamiltonian associated to the program is $H_t \equiv \beta_t v(k_t, c_t) + \lambda_t s(k_t, c_t)$. Maximizing H_t with respect to the costate, state and control variables, gives, respectively: $\partial H_t/\partial \lambda_t = \dot{k}_t$, $\partial H_t/\partial k_t = -\dot{\lambda}_t$, $\partial H_t/\partial c_t = 0$, that is

$$\begin{aligned} \dot{k}_t &= s(k_t, c_t) \\ \dot{\lambda}_t &= -\beta_t \partial v/\partial k_t - \lambda_t \partial s/\partial k_t \\ \frac{\lambda_t}{\beta_t} &= -\frac{\partial v/\partial c_t}{\partial s/\partial c_t} \end{aligned}$$

with $\dot{\beta}_t = -\rho_t \beta_t$ and transversality condition: $\lim_{t \rightarrow \infty} \lambda_t k_t = 0$. Setting $\mu_t \equiv \lambda_t/\beta_t$ and noticing that

$$\dot{\lambda}_t = \beta_t \dot{\mu}_t + \mu_t \dot{\beta}_t \quad (90)$$

we obtain

$$\begin{aligned} \dot{\mu}_t &= -\mu_t \left(\frac{\dot{\beta}_t}{\beta_t} + \frac{\partial s}{\partial k_t} \right) - \frac{\partial v}{\partial k_t} \\ \mu_t &= -\frac{\partial v/\partial c_t}{\partial s/\partial c_t} \end{aligned} \quad (91)$$

¹⁵Functions v and s satisfy the second-order (Arrow-Mangasarian) sufficient conditions for maximization: $\partial^2 s/\partial k^2 < 0$, $(\partial^2 s/\partial k^2)(\partial^2 s/\partial c^2) > [\partial^2 s/(\partial k \partial c)]^2$; $\partial^2 v/\partial k^2 < 0$, $(\partial^2 v/\partial k^2)(\partial^2 v/\partial c^2) > [\partial^2 v/(\partial k \partial c)]^2$.

Applying the Implicit Function Theorem to equation (91) gives $c_t = c(k_t, \mu_t)$ with

$$\frac{\partial c_t}{\partial k_t} = \frac{\frac{\partial v}{\partial c_t} \frac{\partial^2 s}{\partial c_t \partial k_t} - \frac{\partial s}{\partial c_t} \frac{\partial^2 v}{\partial c_t \partial k_t}}{\frac{\partial s}{\partial c_t} \frac{\partial^2 v}{\partial c_t^2} - \frac{\partial v}{\partial c_t} \frac{\partial^2 s}{\partial c_t^2}} \quad (92)$$

$$\frac{\partial c_t}{\partial \mu_t} = \frac{\left(\frac{\partial s}{\partial c_t}\right)^2}{\frac{\partial v}{\partial c_t} \frac{\partial^2 s}{\partial c_t^2} - \frac{\partial s}{\partial c_t} \frac{\partial^2 v}{\partial c_t^2}} \quad (93)$$

Hence, we obtain a two-dimensional system in (k_t, μ_t) :

$$\dot{k}_t = s(k_t, c(k_t, \mu_t)) \quad (94)$$

$$\dot{\mu}_t = -\mu_t \left[\frac{\dot{\beta}_t}{\beta_t} + \frac{\partial s}{\partial k_t}(k_t, c(k_t, \mu_t)) \right] - \frac{\partial v}{\partial k_t}(k_t, c(k_t, \mu_t)) \quad (95)$$

Let, for simplicity, the discount rate be constant over time: $\rho = -\dot{\beta}_t/\beta_t$; at the steady state, the system writes

$$0 = s(k, c(k, \mu)) \quad (96)$$

$$\rho = \frac{\partial s}{\partial k}(k, c(k, \mu)) + \frac{1}{\mu} \frac{\partial v}{\partial k}(k, c(k, \mu)) \quad (97)$$

In order to simplify notation, we will denote first and second-order partial derivatives of a function $z = z(x, y)$ as follows: $(z_x, z_y) \equiv (\partial z/\partial x, \partial z/\partial y)$ and

$$\begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix} \equiv \begin{bmatrix} \partial^2 z/\partial x^2 & \partial^2 z/(\partial x \partial y) \\ \partial^2 z/(\partial y \partial x) & \partial^2 z/\partial y^2 \end{bmatrix}$$

Local dynamics of system (94)-(95) are summarized by the following Jacobian matrix:

$$J_0 \equiv \begin{bmatrix} s_k + s_c c_k & s_c c_\mu \\ -P & \rho - Q \end{bmatrix} \quad (98)$$

where $c_k \equiv \partial c/\partial k$ and $c_\mu \equiv \partial c/\partial \mu$ are given by (92) and (93), and

$$P \equiv v_{kk} + \mu s_{kk} + c_k(v_{kc} + \mu s_{kc}) \quad (99)$$

$$Q \equiv s_k + c_\mu(v_{kc} + \mu s_{kc}) \quad (100)$$

The determinant of the Jacobian matrix is given by

$$T_0 = \rho - Q + s_k + s_c c_k \quad (101)$$

$$D_0 = (\rho - Q)(s_k + s_c c_k) + P s_c c_\mu \quad (102)$$

More explicitly, the trace and the determinant are given by:

$$T_0 = \rho + s_c c_k - c_\mu(v_{kc} + \mu s_{kc})$$

$$D_0 = (\rho - s_k)(s_k + s_c c_k) + c_\mu[s_c(v_{kk} + \mu s_{kk}) - s_k(v_{kc} + \mu s_{kc})]$$

where $\rho - s_k = v_k/\mu$.

This general (reduced) form allows us to derive immediately the stability properties in classical growth models. For instance, we will show that, in the Cass-Koopmans model: $c_k = v_k = v_{kk} = v_{kc} = s_{kc} = 0$ and $c_\mu, s_c, s_{kk} < 0$. Then $T_0 = \rho > 0$ and $D_0 = \mu c_\mu s_c s_{kk} < 0$: the steady state is a saddle point and there is no room for bifurcations. This is a robust property of optimal growth models.

Focus now on the intertemporal optimization model in discrete time.

We maximize the utility series $\sum_{t=0}^{\infty} \beta_t v(k_t, c_t)$ under a sequence of constraints: $k_{t+1} - k_t \leq s(k_t, c_t)$ with $t = 0, 1, \dots$

Under the assumptions $v_c > 0$ and $s_c < 0$, the Lagrangian multipliers are positive and the constraints is binding. The intertemporal smoothing is represented by a sequence of Euler equations. We have the system

$$\begin{aligned} k_{t+1} &= k_t + s(k_t, c_t) \\ \frac{\mu_t}{\mu_{t+1}} &= \frac{\beta_{t+1}}{\beta_t} \left[1 + \frac{\partial s}{\partial k_{t+1}}(k_{t+1}, c_{t+1}) + \frac{1}{\mu_{t+1}} \frac{\partial v}{\partial k_{t+1}}(k_{t+1}, c_{t+1}) \right] \end{aligned}$$

where μ_t is still given by (91). As above (91) allows us to define $c_t = c(k_t, \mu_t)$ with partial derivatives (92) and (93).

Discrete-time dynamics are eventually given by

$$k_{t+1} = k_t + s(k_t, c(k_t, \mu_t)) \quad (103)$$

$$\begin{aligned} &\frac{\mu_t}{\mu_{t+1}} \\ &= \frac{\beta_{t+1}}{\beta_t} \left[1 + \frac{\partial s}{\partial k_{t+1}}(k_{t+1}, c(k_{t+1}, \mu_{t+1})) + \frac{1}{\mu_{t+1}} \frac{\partial v}{\partial k_{t+1}}(k_{t+1}, c(k_{t+1}, \mu_{t+1})) \right] \end{aligned} \quad (104)$$

that is a two-dimensional system in the pair of variables (k_t, μ_t) . k_t is a state variable, while μ_t is a jump variable. Notice also that μ_t is the current-value costate variable of the continuous-time program at time t , that is $\lambda_t = \beta_t \mu_t$.

5.1.2 Discretized optimization models

In this section, the main question we tackle is whether the discrete time system (103)-(104) can be recovered through a (first-order) Euler discretization.

We mix a backward-looking discretization of constraint (94) and a forward-looking discretization of the Euler equation (95).

Discretizing the continuous-time constraint (94) gives:

$$k_{t+h} - k_t \approx h s(k_t, c(k_t, \mu_t)) \quad (105)$$

that is the discrete-time resource constraint (103) under a unit discretization step ($h = 1$).

Because of the forward-looking nature of the Euler equation, we can not recover (104) in backward-looking. Using (90), equation (95) can be written in terms of $\lambda_t \equiv \beta_t \mu_t$ instead of μ_t :

$$\dot{\lambda}_t = -\lambda_t \frac{\partial s}{\partial k_t} \left(k_t, c \left(k_t, \frac{\lambda_t}{\beta_t} \right) \right) - \beta_t \frac{\partial v}{\partial k_t} \left(k_t, c \left(k_t, \frac{\lambda_t}{\beta_t} \right) \right) \quad (106)$$

Let us call (106) a λ -type Euler equation and apply the forward-looking discretization (3) to (106):

$$\begin{aligned} & \lambda_{t+h} - \lambda_t \\ = & -h \left[\lambda_{t+h} \frac{\partial s}{\partial k_{t+h}} \left(k_{t+h}, c \left(k_{t+h}, \frac{\lambda_{t+h}}{\beta_{t+h}} \right) \right) + \beta_{t+h} \frac{\partial v}{\partial k_{t+h}} \left(k_{t+h}, c \left(k_{t+h}, \frac{\lambda_{t+h}}{\beta_{t+h}} \right) \right) \right] \end{aligned}$$

Replacing $\lambda_t = \beta_t \mu_t$, we obtain

$$\begin{aligned} & \frac{\beta_t}{\beta_{t+h}} \frac{\mu_t}{\mu_{t+h}} \\ = & 1 + h \left[\frac{\partial s}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) + \frac{1}{\mu_{t+h}} \frac{\partial v}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) \right] \end{aligned} \quad (107)$$

that is the discrete-time Euler equation (104) under a unit discretization step $h = 1$.

We notice that, under no discounting, as in the original Ramsey (1928), considering λ or μ is indifferent. Conversely, now the choice of multiplier matters: only the discretization of a λ -type Euler equation allows us to recover the discrete-time model. Discretizing a μ -type equation gives another approximation which is still right but different from the usual one.

We conclude that traditional growth models in discrete time come from a unit step hybrid approximation of the continuous-time system: backward-looking discretization of the constraint and a forward-looking discretization of a λ -type Euler equation.

In the case of the Solow model, we have studied the dynamic properties of a quadratic discretization and, in particular, the surprising impossibility of flip bifurcations. Along these lines, we can derive a quadratic approximation of (94)-(106), now hybrid because of the forward-looking nature of consumption smoothing. More precisely, we apply discretization (7) to (94) and (14) to (106).

$$\begin{aligned} \dot{k}_t &= s \left(k_t, c \left(k_t, \frac{\lambda_t}{\beta_t} \right) \right) \equiv f_1(k_t, \lambda_t) \\ \dot{\lambda}_t &= -\lambda_t \frac{\partial s}{\partial k_t} \left(k_t, c \left(k_t, \frac{\lambda_t}{\beta_t} \right) \right) - \beta_t \frac{\partial v}{\partial k_t} \left(k_t, c \left(k_t, \frac{\lambda_t}{\beta_t} \right) \right) \equiv f_2(k_t, \lambda_t) \end{aligned}$$

become, respectively:

$$\begin{aligned}
k_{t+h} - k_t &\approx h f_1(k_t, \lambda_t) \\
&\quad + \frac{h^2}{2} \left[f_1(k_t, \lambda_t) \frac{\partial f_1(k_t, \lambda_t)}{\partial k_t} + f_2(k_t, \lambda_t) \frac{\partial f_1(k_t, \lambda_t)}{\partial \lambda_t} \right] \\
\lambda_{t+h} - \lambda_t &\approx h f_2(k_{t+h}, \lambda_{t+h}) \\
&\quad - \frac{h^2}{2} \left[f_1(k_{t+h}, \lambda_{t+h}) \frac{\partial f_2(k_{t+h}, \lambda_{t+h})}{\partial k_{t+h}} + f_2(k_{t+h}, \lambda_{t+h}) \frac{\partial f_2(k_{t+h}, \lambda_{t+h})}{\partial \lambda_{t+h}} \right]
\end{aligned}$$

or, more explicitly:

$$k_{t+h} - k_t \approx h s_t + \frac{h^2}{2} \left(s_t - \mu_t q_{1t} \frac{\partial s}{\partial c_t} \frac{\partial c}{\partial \mu_t} \right) \frac{\partial s}{\partial k_t} \quad (108)$$

$$\begin{aligned}
\frac{\beta_t}{\beta_{t+h}} \frac{\mu_t}{\mu_{t+h}} &\approx 1 + h q_{1t+h} \frac{\partial s}{\partial k_{t+h}} \\
&\quad + \frac{h^2}{2} q_{1t+h} \frac{\partial s}{\partial k_{t+h}} \left(\frac{\partial s}{\partial k_{t+h}} + \mu_{t+h} q_{3t+h} \frac{\partial c}{\partial \mu_{t+h}} \frac{\partial^2 s}{\partial k_{t+h} \partial c_{t+h}} \right) \\
&\quad - \frac{h^2}{2} s_{t+h} \left(q_{2t+h} \frac{\partial^2 s}{\partial k_{t+h}^2} + q_{3t+h} \frac{\partial c}{\partial k_{t+h}} \frac{\partial^2 s}{\partial k_{t+h} \partial c_{t+h}} \right) \quad (109)
\end{aligned}$$

where $s_t \equiv s(k_t, c(k_t, \mu_t))$ and

$$\begin{aligned}
q_{1t} &\equiv 1 + \frac{1}{\mu_t} \frac{\partial v / \partial k_t}{\partial s / \partial k_t} \\
q_{2t} &\equiv 1 + \frac{1}{\mu_t} \frac{\partial^2 v / \partial k_t^2}{\partial^2 s / \partial k_t^2} \\
q_{3t} &\equiv 1 + \frac{1}{\mu_t} \frac{\partial^2 v / (\partial k_t \partial c_t)}{\partial^2 s / (\partial k_t \partial c_t)}
\end{aligned}$$

We observe that in the Cass-Koopmans model (and *a fortiori* in Ramsey) the utility function no longer depends on capital: $v(k_t, c_t) = u(c_t)$, and, therefore, $q_{1t} = q_{2t} = q_{3t}$. System (108)-(109) reduces to:

$$k_{t+h} - k_t \approx h s_t + \frac{h^2}{2} \left(s_t - \mu_t \frac{\partial s}{\partial c_t} \frac{\partial c}{\partial \mu_t} \right) \frac{\partial s}{\partial k_t} \quad (110)$$

$$\begin{aligned}
\frac{\beta_t}{\beta_{t+h}} \frac{\mu_t}{\mu_{t+h}} &\approx 1 + h \frac{\partial s}{\partial k_{t+h}} + \frac{h^2}{2} \frac{\partial s}{\partial k_{t+h}} \left(\frac{\partial s}{\partial k_{t+h}} + \mu_{t+h} \frac{\partial c}{\partial \mu_{t+h}} \frac{\partial^2 s}{\partial k_{t+h} \partial c_{t+h}} \right) \\
&\quad - \frac{h^2}{2} s_{t+h} \left(\frac{\partial^2 s}{\partial k_{t+h}^2} + \frac{\partial c}{\partial k_{t+h}} \frac{\partial^2 s}{\partial k_{t+h} \partial c_{t+h}} \right) \quad (111)
\end{aligned}$$

In order to study and compare elementary bifurcations in discrete and continuous time, focus now on linear discretizations.¹⁶

¹⁶We omit, for brevity, the linearization of quadratic discretizations.

Discretizing $\dot{\beta}_t$ in forward-looking, we obtain

$$\beta_{t+h}/\beta_t \approx 1/(1+h\rho_{t+h}) \quad (112)$$

We can replace (112) in (107) to get

$$\frac{\mu_t}{\mu_{t+h}} = \frac{1+h \left[\frac{\partial s}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) + \frac{1}{\mu_{t+h}} \frac{\partial v}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) \right]}{1+h\rho_{t+h}} \quad (113)$$

The existence of a steady state requires $\rho_t = \rho$ constant over time. A constant discounting implies $\beta_{t+h}/\beta_t = \beta^h$, where $\beta = e^{-\rho}$. In this case, at the steady state, (113) gives (97). Assumption 4 on the fundamentals ensures the existence and the uniqueness of the steady state given by (96)-(97).

Focus now the local dynamics. Since, at the steady state, μ_t is stationary (while $\lambda_t = \beta_t \mu_t$ is not because β_t decreases over time), we linearize the system with a forward-looking μ -type Euler discretization.

The hybrid Euler discretization (105)-(107) becomes

$$k_{t+h} \approx k_t + h s(k_t, c(k_t, \mu_t)) \quad (114)$$

$$\begin{aligned} & (1+h\rho) \frac{\mu_t}{\mu_{t+h}} \\ = & 1+h \left[\frac{\partial s}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) + \frac{1}{\mu_{t+h}} \frac{\partial v}{\partial k_{t+h}} (k_{t+h}, c(k_{t+h}, \mu_{t+h})) \right] \end{aligned} \quad (115)$$

We linearize (114)-(115) to obtain

$$dk_{t+h} = [1+h(s_k + s_c c_k)] dk_t + h s_c c_\mu d\mu_t$$

and

$$\begin{aligned} & h[v_{kk} + \mu s_{kk} + c_k(v_{kc} + \mu s_{kc})] dk_{t+h} \\ & + (1+h[\rho - v_k/\mu + c_\mu(v_{kc} + \mu s_{kc})]) d\mu_{t+h} \\ = & (1+h\rho) d\mu_t \end{aligned}$$

that is,

$$\begin{aligned} & h[v_{kk} + \mu s_{kk} + c_k(v_{kc} + \mu s_{kc})] dk_{t+h} + (1+h[s_k + c_\mu(v_{kc} + \mu s_{kc})]) d\mu_{t+h} \\ = & (1+h\rho) d\mu_t \end{aligned}$$

(notice from (97) that $s_k = \rho - v_k/\mu$). Using (99) and (100), we find the associated Jacobian matrix J_1 :

$$J_1 \equiv \begin{bmatrix} 1+h(s_k + s_c c_k) & h s_c c_\mu \\ -[1+h(s_k + s_c c_k)] \frac{hP}{1+hQ} & \frac{1+h\rho}{1+hQ} - \frac{hP}{1+hQ} h s_c c_\mu \end{bmatrix} \quad (116)$$

with the following trace and determinant

$$D_1 = [1 + h(s_k + s_c c_k)] \frac{1 + h\rho}{1 + hQ} \quad (117)$$

$$T_1 = 1 + h(s_k + s_c c_k) + \frac{1 + h\rho}{1 + Qh} - \frac{Ph}{1 + Qh} h s_c c_\mu \quad (118)$$

A saddle-node bifurcation requires $D_0 = 0$ in continuous time and $D_1 = T_1 - 1$ in discrete time. It is easy to check that $h^2 D_0 = (1 + Qh)(D_1 - T_1 + 1)$, where D_0 is given by (102) and T_1 and D_1 are given by (117) and (118). Therefore $D_0 = 0$ iff $D_1 = T_1 - 1$. This means that a saddle-node bifurcation generically arises in continuous time if and only if it occurs in discrete time.

A Hopf bifurcation in continuous time generically requires $T_0 = 0$ and $D_0 > 0$. In discrete time, a Hopf bifurcation needs $D_1 = 1$ and $T_1^2 \leq 4$. We observe that

$$T_1 = 2 + \frac{h[T_0(1 + h\rho) - hD_0 - h\rho(\rho - Q)]}{1 + h\rho - h(\rho - Q)} \quad (119)$$

$$D_1 = 1 + \frac{h[T_0(1 + h\rho) - h\rho(\rho - Q)]}{1 + h\rho - h(\rho - Q)} \quad (120)$$

where T_0 , D_0 , T_1 , D_1 are respectively given by (101), (102), (117) and (118).

According to (97) and (100), when v no longer depends on k (as in the Ramsey-Cass-Koopmans framework), we have $\rho = Q$. Then (119) and (120) reduce to¹⁷

$$T_1 = 1 + D_1 - \frac{h^2}{1 + h\rho} D_0 \quad (121)$$

$$D_1 = 1 + hT_0 \quad (122)$$

and $T_0 = 0$ iff $D_1 = 1$. Using (121) with $D_0 > 0$ and $D_1 = 1$, condition $T_1^2 \leq 4$ is equivalent to $h \leq 2 \left[\rho/D_0 + \sqrt{1 + (\rho/D_0)^2} \right]$.

Therefore, if a Hopf bifurcation arises in continuous time, under a sufficiently small discretization step, it occurs generically also in discrete time.

5.2 Ramsey models

The most popular optimal growth model is undoubtedly Ramsey (1928), later refined by Cass (1965) and Koopmans (1965). Following the original Ramsey (1928), we assume no discounting: $\rho = 0$. Ramsey argued against discounting utility of future generations as being "ethically indefensible". Cass and Koopmans introduced discounting and compared the market equilibrium with the planner's solution.

In the original Ramsey model, a social planner maximizes a dynastic utility by choosing the intertemporal allocation of consumption. In contrast to

¹⁷We observe that, when $\rho = 0$, expressions (121)-(122) reduce to (29)-(30) in footnote 6.

Kaldor (1940) and Solow (1956), Ramsey-type models are characterized by an intertemporal utility maximization resulting in an endogenous saving rate. In the following, to keep things as simple as possible, we assume no population growth ($n = 0$).

5.2.1 Ramsey models

At first, we consider a continuous-time dynamics. A benevolent planner chooses the consumption path and the profile of capital accumulation in order to maximize the representative consumer's utility functional (87) subject to the resource constraint (88) with the following specification of the fundamentals:

$$s(k_t, c_t) = f(k_t) - \delta k_t - c_t \quad (123)$$

$$v(k_t, c_t) = u(c_t) - u(c) \quad (124)$$

and $\beta_t \equiv 1$ for every $t \geq 0$.

The instantaneous utility is defined by $u(c_t) - u(c)$, where c denotes the "bliss point". In order to ensure the utility functional to be bounded (the integral to converge) we fix a particular bliss point value: $c = f(k) - \delta k$ with $f'(k) = \delta$. This bliss point is the steady state value of consumption in the Ramsey model.¹⁸

Assumption 5 *The intensive production function $f(k)$ is C^2 , strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$) in the capital intensity and satisfies the Inada conditions.*

Assumption 6 *The instantaneous utility function u is C^2 , strictly increasing ($u'(c) > 0$) and strictly concave ($u''(c) < 0$) in the consumption level. $\varepsilon(c) \equiv -u'(c) / [u''(c)c] > 0$ will denote the elasticity of intertemporal substitution.*

Equation (91) reduces to $\mu = u'(c)$ and $c = c(k, \mu) = u'^{-1}(\mu) \equiv d(\mu)$ with $\partial c / \partial k = 0$ and $\partial c / \partial \mu = 1 / u''$. Assumption 6 (namely, the strict concavity of u) entails that c_t is a well-defined function of μ_t . System (94)-(95) simplifies to:

$$\dot{k}_t = f(k_t) - \delta k_t - d(\mu_t) \quad (125)$$

$$\dot{\mu}_t = -\mu_t [f'(k_t) - \delta] \quad (126)$$

The transversality condition of discounted programs no longer applies. Moreover, all the model of dynamic optimization where the transversality condition (as a necessary condition for maximization) plays no role, are those without discounting (Pitchford (1977)).

Focus now on discrete-time dynamics. The planner maximizes the intertemporal utility $\sum_{t=0}^{\infty} [u(c_t) - u(c)]$ subject to the sequence of resource constraints: $k_{t+1} - k_t + c_t \leq f(k_t) - \delta k_t$. c still denotes the bliss point.

¹⁸ Compare the Ramsey (1928) and Cass-Koopmans (1965): the bliss point value is nothing else than the modified golden rule under a null discount rate.

System (103)-104) writes:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - d(\mu_t) \quad (127)$$

$$\frac{\mu_t}{\mu_{t+1}} = 1 + f'(k_{t+1}) - \delta \quad (128)$$

5.2.2 Discretized Ramsey models

The issue we address is whether system (127)-(128) can be obtained as an Euler discretization of the continuous-time system. The answer is positive under an appropriate hybrid discretization. As seen above, the added value of Ramsey with respect to Solow is the intertemporal utility maximization; the resource constraint remains the same, but the Euler equation replaces the exogenous saving rate. We have shown that the discrete-time Solow model can be obtained through a linear discretization.¹⁹

Under (123) and (124), system (114)-(115) writes

$$k_{t+h} - k_t \approx h [f(k_t) - \delta k_t - d(\mu_t)] \quad (129)$$

$$\frac{\mu_t}{\mu_{t+h}} \approx 1 + h [f'(k_{t+h}) - \delta] \quad (130)$$

and, setting a unit discretization step ($h = 1$), we recover exactly the discrete-time system (127)-(128).

Thus, the discrete-time Ramsey model comes from a first-order hybrid discretization of the continuous-time model, that is a backward-looking discretization of the resource constraint (125) (apply (2)) and a forward-looking discretization of the Euler equation (126) (apply (3)).

On the one side, the resource constraint is backward-looking because the capital stock is a state variable; on the other side, the consumption smoothing (Euler equation) rests on current saving decisions that depend on the expected interest rate. These arguments account for a hybrid discretization.

Focus on the steady state. Assumption 5 and Assumption 6 ensure the existence and uniqueness of the steady state. Equations (96) and (97) become respectively: $c = f(k) - \delta k$ and $f'(k) = \delta$: the steady state is the same for dynamic systems (125)-(126), (127)-(128) and (129)-(130).

Focus now on local dynamics. Let

$$\alpha(k) \equiv k f'(k) / f(k) \quad (131)$$

$$\varepsilon_r(k) \equiv k f''(k) / f'(k) = -[1 - \alpha(k)] / \sigma(k) < 0 \quad (132)$$

be the capital-share in total income and the elasticity of interest rate. Let $\sigma(k)$ and $\varepsilon(c)$ denote the elasticities of capital-labor substitution and of intertemporal substitution.

The Jacobian matrix of the continuous-time system (98) reduces to:

$$J_0 = \begin{bmatrix} 0 & -1/u''(c) \\ -u'(c) f''(k) & 0 \end{bmatrix} = \begin{bmatrix} 0 & Ak/\mu \\ B\mu/k & 0 \end{bmatrix} \quad (133)$$

¹⁹ More precisely, if $h = 1$, system (56)-(57) reduces to (52)-(53).

where $A \equiv \varepsilon\delta(1-\alpha)/\alpha > 0$ and $B \equiv \delta(1-\alpha)/\sigma > 0$. The trace and the determinant become $T_0 = 0$ and $D_0 = -AB < 0$: the steady state is a saddle point and there is no room for bifurcations.

Focus now on the hybrid discretization (129)-(130). We can apply and simplify the reduced forms of the general case to obtain the following proposition:

Proposition 13 (*Ramsey*) *The stability properties of the hybrid discretization are consistent with those of the continuous-time model. More precisely, (1) the hybrid trace and determinant are $T_1 = 2 + h^2AB$ and $D_1 = 1 < T_1 - 1$, (2) the hybrid discretization is saddle-point stable (as the continuous-time system), whatever the discretization step h .*

Proof. The Jacobian matrix J_1 of the hybrid Euler discretization (129)-(130) is obtained from (116) using (123) and (124):

$$J_1 \equiv \begin{bmatrix} 1 & 0 \\ h\mu f''(k) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -h/u''(c) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & hA\frac{k}{\mu} \\ hB\frac{\mu}{k} & 1 + h^2AB \end{bmatrix}$$

with A and B as above. Trace and determinant (117) and (118) simplify: $T_1 = 2 + h^2AB = 2 - h^2D_0$ and $D_1 = 1$. Since $D_1 = 1$ and $D_1 < T_1 - 1$, the pair (T_1, D_1) lies in the cone of the saddle points, whatever the discretization step h . There is no room for bifurcations, as in the continuous-time case. ■

5.3 Cass-Koopmans models

The "ethical" undiscounted utility functional in Ramsey (1928) is replaced in the Cass-Koopmans model (1965) by a weighted average of future felicities with decreasing weights over time (discounting).

5.3.1 Cass-Koopmans models

A benevolent planner determines the profile of capital accumulation in order to maximize the representative consumer's utility functional (87) subject to the resource constraint (88).²⁰ In Cass (1965) and Koopmans (1965), saving and utility functions are specified as

$$s(k_t, c_t) = f(k_t) - \delta k_t - c_t \quad (134)$$

$$v(k_t, c_t) = u(c_t) \quad (135)$$

and satisfy Assumptions 5 and 6. We assume no population growth. As in the Ramsey model, equation (91) reduces to $c = d(\mu)$ with $\mu = u'(c)$ and $d'(\mu) = 1/u''$. System (94)-(95) simplifies to:

$$\dot{k}_t = f(k_t) - \delta k_t - d(\mu_t) \quad (136)$$

$$\dot{\mu}_t = \mu_t [\rho_t + \delta - f'(k_t)] \quad (137)$$

²⁰Without imperfections, the market economy decentralizes the benevolent planner's solution (first welfare theorem).

In discrete time, the planner maximizes the utility series $\sum_{t=0}^{\infty} \beta_t u(c_t)$ subject to the sequence of resource constraints: $k_{t+1} - k_t + c_t \leq f(k_t) - \delta k_t$ with $t = 0, 1 \dots$. Under fundamentals (134) and (135), system (103)-(104) writes:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \quad (138)$$

$$\frac{\mu_t}{\mu_{t+1}} = \frac{\beta_{t+1}}{\beta_t} [1 + f'(k_{t+1}) - \delta] \quad (139)$$

5.3.2 Discretized Cass-Koopmans models

Saving decisions are forward-looking: under an appropriate hybrid discretization, that is, a backward-looking discretization of the resource constraint (apply (2) as in the Solow model) and a forward-looking discretization of the Euler equation (apply (3) as in the Ramsey model), we recover the traditional discrete-time Cass-Koopmans model.

Under (134) and (135), system (114)-(115) reduces to

$$\begin{aligned} k_{t+h} - k_t &\approx h[f(k_t) - \delta k_t - d(\mu_t)] \\ \frac{\mu_t}{\mu_{t+h}} &\approx \frac{\beta_{t+h}}{\beta_t} (1 + h[f'(k_{t+h}) - \delta]) \end{aligned} \quad (140)$$

and becomes the discrete-time system (138)-(139) with $h = 1$ (unit discretization step).

We notice that, while in the Ramsey model considering λ or μ was indifferent because of no discounting, now the choice of multiplier matters: only the discretization of a λ -type Euler equation allows us to recover the traditional discrete-time model. Discretizing a μ -type equation gives another approximation which is also right, but different from the usual Euler equation in discrete time.²¹

²¹Indeed, the reader must be aware that the discrete-time Cass-Koopmans model is only one of the possible first-order discretizations of the continuous-time system (136)-(137).

For simplicity, focus only on the Euler equation and omit the backward-looking approximation of the resource constraint. (137) can be equivalently written as $\dot{x}_t = \rho_t + \delta - f'(k_t)$ with $x_t \equiv \ln \mu_t$.

Focus first on the forward-looking. Discretizing $\dot{\beta}_t$ or $\dot{\beta}_t/\beta_t$ gives, respectively, $\beta_{t+h}/\beta_t \approx 1/(1 + h\rho_{t+h})$ or $\beta_{t+h}/\beta_t \approx e^{-h\rho_{t+h}}$. If $\rho_t = \rho$ is constant over time, we can define the discount factor with a unit step without caring about discretizing in backward or forward-looking: $\beta \equiv \beta_{t+1}/\beta_t \approx e^{-\rho}$.

Using the appropriate discounting discretization and linearizing $\dot{\mu}_t$ or \dot{x}_t gives, respectively,

$$\frac{\mu_t}{\mu_{t+h}} \approx 1 + h[f'(k_{t+h}) - \delta] - \frac{\beta_t - \beta_{t+h}}{\beta_{t+h}} \quad (141)$$

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{\beta_{t+h}}{\beta_t} e^{h[f'(k_{t+h}) - \delta]} \quad (142)$$

The sense of such approximations can be understood, by considering, for instance, (142). We know that $e^z \approx 1+z$, when z is close to zero. Replacing $e^{h[f'(k_{t+h}) - \delta]}$ with $1 + h[f'(k_{t+h}) - \delta]$ gives, exactly, (140), that is (139).

In backward-looking, discretizing $\dot{\beta}_t$ or $\dot{\beta}_t/\beta_t$ gives, respectively, $\beta_{t+h}/\beta_t \approx 1 - h\rho_t$ or $\beta_{t+h}/\beta_t \approx e^{-h\rho_t}$. Using the opportune discounting discretization and approximating, for

Let us now focus on a second-order hybrid discretization. Discretization (110)-(111) comes from a backward-looking second-order discretization of the resource constraint and a forward-looking second-order discretization of the λ -type Euler equation. Replacing (134) and (135) in discretization (110)-(111), we find a well-defined hybrid system in $(k_t, \mu_t, k_{t+h}, \mu_{t+h})$:

$$\begin{aligned} k_{t+h} - k_t &\approx h [f(k_t) - \delta k_t - d(\mu_t)] \\ &\quad + \frac{h^2}{2} [f'(k_t) - \delta] [f(k_t) - \delta k_t - d(\mu_t) [1 + \varepsilon(d(\mu_t))]] \end{aligned} \quad (146)$$

$$\begin{aligned} \frac{\mu_t}{\mu_{t+h}} &\approx \frac{\beta_{t+h}}{\beta_t} [1 + h [f'(k_{t+h}) - \delta] \\ &\quad + \frac{h^2}{2} ([f'(k_{t+h}) - \delta]^2 - [f(k_{t+h}) - \delta k_{t+h} - d(\mu_{t+h})] f''(k_{t+h}))] \end{aligned} \quad (147)$$

Under a unit discretization step, we eventually find the second-order Cass-Koopmans model in discrete time, which refines the traditional model (138)-(139):

$$\begin{aligned} k_{t+1} - k_t &\approx f(k_t) - \delta k_t - d(\mu_t) \\ &\quad + \frac{1}{2} [f'(k_t) - \delta] [f(k_t) - \delta k_t - d(\mu_t) [1 + \varepsilon(d(\mu_t))]] \\ \frac{\mu_t}{\mu_{t+1}} &\approx \frac{\beta_{t+1}}{\beta_t} [1 + f'(k_{t+1}) - \delta \\ &\quad + \frac{1}{2} ([f'(k_{t+1}) - \delta]^2 - [f(k_{t+1}) - \delta k_{t+1} - d(\mu_{t+1})] f''(k_{t+1}))] \end{aligned}$$

For simplicity, we omit the linearization of quadratic discretizations and we focus on linear discretizations to compare elementary bifurcations in discrete and continuous time.

Assume a constant discounting: $\rho_t = \rho$ (that is, $\beta_{t+h}/\beta_t = \beta^h$ with $\beta = e^{-\rho}$). Under Assumption 5 and 6, the steady state is unique. Equations (96)

instance, $\dot{\lambda}_t$, $\dot{\mu}_t$ or \dot{x}_t implies, respectively,

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{\beta_{t+h}}{\beta_t} \frac{1}{1 - h [f'(k_t) - \delta]} \quad (143)$$

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{1}{1 - h [f'(k_t) - \delta] + \frac{\beta_t - \beta_{t+h}}{\beta_t}} \quad (144)$$

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{\beta_{t+h}}{\beta_t} e^{h[f'(k_t) - \delta]} \quad (145)$$

Dynamics generated by backward-looking approximations are very different from (139) because the productivity depends on k_t instead of k_{t+1} . Forward-looking dynamics are more appropriate to capture saving decisions that depend on future returns.

and (97) give the modified golden rule: $c = f(k) - \delta k$ and²²

$$\rho = f'(k) - \delta \quad (149)$$

Focus now on local dynamics. We linearize the system (136)-(137) around (k, μ) (as seen above, μ is stationary, while $\lambda_t = \beta_t \mu$ is not).

The continuous-time Jacobian matrix (98) reduces to:

$$J_0 \equiv \begin{bmatrix} \rho & -\frac{1}{u''(c)} \\ -u'(c)f''(k) & 0 \end{bmatrix} = \begin{bmatrix} \rho & Ak/\mu \\ B\mu/k & 0 \end{bmatrix} \quad (150)$$

where $A \equiv \varepsilon [\rho/\alpha + \delta(1-\alpha)/\alpha] > 0$ and $B \equiv (\rho + \delta)(1-\alpha)/\sigma > 0$.

Matrix (150) generalizes (133) with discounting ($\rho > 0$): under no discounting ($\rho = 0$), the Cass-Koopmans model collapses in the Ramsey model. As above, α , σ and ε denote, respectively, the capital-share in total income, the elasticity of capital-labor substitution and the elasticity of intertemporal substitution (see also formulas (131)-(132)).

The trace and the determinant in continuous time are $T_0 = \rho > 0$ and $D_0 = -AB < 0$ and the steady state is a saddle point: there is no room for bifurcations.

Focus now on the hybrid Euler discretization (129)-(130) and the Jacobian matrix J_1 obtained from (116) specifying the fundamentals as in (134) and (135):

$$J_1 \equiv \begin{bmatrix} 1 + h\rho & hAk/\mu \\ hB\mu/k & 1 + ABh^2/(1 + h\rho) \end{bmatrix} \quad (151)$$

The determinant and the trace of the hybrid system are $D_1 = 1 + h\rho = \beta_t/\beta_{t+h} = \beta^{-h}$ and $T_1 = 1 + D_1 + h^2AB/(1 + h\rho)$.²³

Since $D_1 > 1$ and $D_1 = T_1 - 1 - h^2AB/(1 + h\rho) < T_1 - 1$, the steady state is a saddle point, whatever the discretization step h . There is no room for bifurcations, as in the continuous-time case.

5.4 Cass-Koopmans models with externalities

Externalities can affect either the production or the utility levels of economic agents. The public goods constitute a prominent class of externalities. Zhang (2000) introduces externalities of public spending in the Cass-Koopmans framework. As in Barro (1990), the public good plays the role of positive productive externality. However, Zhang (2000) considers also a public consumption good

²²Discretizing $\dot{\beta}_t$ in forward-looking gives

$$\beta_{t+h}/\beta_t \approx 1/(1 + h\rho_{t+h}) \quad (148)$$

while, replacing (148) in (140) gives

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{1 + h[f'(k_{t+h}) - \delta]}{1 + h\rho_{t+h}}$$

that is, at the steady state, (149).

²³We observe that, when $h = 1$, then $D_1 = 1/\beta$.

which enters households' utility functions. In his original model, Cobb-Douglas technology and preferences are considered and time is continuous.

We generalize Zhang in two directions: on the one side, we use more general production and utility functions; on the other side, we provide a discrete-time version of Zhang and we compare bifurcations in continuous and discrete time.

Exemplifying one of the simplest Hopf bifurcations in a two-dimensional economy is the main asset of Zhang (2000) and the very sense of considering his model in our work.

5.4.1 Zhang models

Zhang (2000) introduces two externalities in the Cass-Koopmans model.

(1) On the hand, there are positive externalities of public capital in a homogeneous production function as in Barro (1990): $Y \equiv F(K, L, g)$ or, in intensive terms, $y = f(k, g)$, where $y \equiv Y/L$ and $k \equiv K/L$.

(2) On the other hand, we introduce positive externality of public capital in the utility function: $u_t = u(c_t, g_t)$.

These functions fulfill the following properties.

Assumption 7 *The production function F is CRS with respect to (K_t, L_t) . The intensive production function $f(k, g)$ is C^2 , increasing in k and g and strictly concave in the private capital k ($\partial f / \partial k > 0$ and $\partial^2 f / \partial k^2 < 0$). In addition: $\partial^2 f / (\partial g \partial k) > 0$.*

Assumption 8 *The utility function u is C^2 , strictly increasing in c and g ($\partial u / \partial c > 0$, $\partial u / \partial g > 0$) and strictly concave in c ($\partial^2 u / \partial c^2 < 0$).*

According to Assumption 7, the impact of public capital on private production is positive ($\partial f / \partial g > 0$) and positively affects the marginal productivity of private capital ($\partial^2 f / (\partial g \partial k) > 0$).

For simplicity, we assume no population growth ($n = 0$) and no capital depreciation ($\delta = 0$).

The public budget is assumed to be balanced over time and the receipts to come from a homogenous tax on labor and capital earnings: $G_t = \tau Y_t = \tau F(K_t, L_t, g_t)$ (or, in per capita terms, $g_t = \tau y_t = \tau f(k_t, g_t)$). The implicit equation $g_t = \tau f(k_t, g_t)$ locally determines the equilibrium public spending as a function of capital stock:

$$g_t = g(k_t) \tag{152}$$

Assumption 9

$$dg/dk = (\tau \partial f / \partial k) / (1 - \tau \partial f / \partial g) > 0 \tag{153}$$

In the following, we focus on the competitive dynamics.²⁴ The representative household chooses the consumption path and the profile of capital accumulation in order to maximize the utility functional (87) subject to the resource constraint (88) with a new specification of fundamentals:

$$s(k_t, c_t) = (1 - \tau)(r_t k_t + w_t l_t) - c_t \quad (154)$$

$$v(k_t, c_t) = u(c_t, g_t) \quad (155)$$

The initial endowment k_0 is given.

Under Assumption 7, profit maximization gives

$$(r_t, w_t) = (\partial f / \partial k_t, f(k_t, g_t) - k_t \partial f / \partial k_t) \quad (156)$$

while $g_t = g(k_t)$ solves the government budget constraint:

$$g_t = \tau f(k_t, g_t) \quad (157)$$

For simplicity, we assume an inelastic labor supply: $l_t = 1$.

Under Assumptions 7 and 8 that replace Assumption 4, we can substitute (156) in the dynamic system (94)-(95) to obtain:²⁵

$$\dot{k}_t = (1 - \tau) f(k_t, g(k_t)) - c(k_t, \mu_t) \quad (158)$$

$$\dot{\mu}_t = \mu_t \left[\rho_t - (1 - \tau) \frac{\partial f}{\partial k_t}(k_t, g(k_t)) \right] \quad (159)$$

In discrete time, the households maximize the utility series $\sum_{t=0}^{\infty} \beta_t u(c_t, g_t)$ subject to the sequence of budget constraints: $k_{t+1} - k_t + c_t \leq (1 - \tau)(r_t k_t + w_t l_t)$ with $t = 0, 1, \dots$

With fundamentals (154) and (155), system (103)-104) reduces to

$$k_{t+1} - k_t = (1 - \tau)(r_t k_t + w_t l_t) - c_t \quad (160)$$

$$\frac{\mu_t}{\mu_{t+1}} = \frac{\beta_{t+1}}{\beta_t} [1 + (1 - \tau) r_{t+1}] \quad (161)$$

Substituting $l_t = 1$ and (156) in (160)-(161), one gets

$$k_{t+1} - k_t = (1 - \tau) f(k_t, g(k_t)) - c(\mu_t, g_t(k_t)) \quad (162)$$

$$\frac{\mu_t}{\mu_{t+1}} = \frac{\beta_{t+1}}{\beta_t} \left[1 + (1 - \tau) \frac{\partial f}{\partial k_{t+1}}(k_{t+1}, g(k_{t+1})) \right] \quad (163)$$

where $\mu_t = \partial u / \partial c_t$ is the current-value costate variable of the continuous-time program.

²⁴ Because of the externalities, the benevolent planner's solution differs from the competitive market solution (the planner internalizes the external effects).

²⁵ The capital stock enters indirectly the utility function through the public good, which is an externality. Since we consider a market economy, households maximizes the utility functional taking g as given, and the (Arrow-Mangasarian) second-order conditions reduce to the partial concavity of u ($\partial^2 u / \partial c^2 < 0$) jointly with the partial concavity of s ($\partial^2 f / \partial k^2 < 0$).

5.4.2 Discretized Zhang models

The issue we tackle, is still whether discrete-time dynamics can be obtained through an Euler discretization of the continuous-time system.

As above, the answer is positive if we choose a hybrid discretization, that is, backward and forward-looking discretizations for the resource constraint and the Euler equation, respectively.

Under (154) and (155), system (114)-(115) simplifies to

$$k_{t+h} - k_t \approx h[(1 - \tau)f(k_t, g(k_t)) - c(\mu_t, g_t(k_t))] \quad (164)$$

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{\beta_{t+h}}{\beta_t} \left[1 + h(1 - \tau) \frac{\partial f}{\partial k_{t+h}}(k_{t+h}, g(k_{t+h})) \right] \quad (165)$$

and, setting a unit discretization step ($h = 1$), we recover the discrete-time system (162)-(163).

As was the case in the Cass-Koopmans model, the discrete-time Zhang model comes from a backward-looking discretization of the resource constraint and a forward-looking discretization of a λ -type Euler equation, what we call a hybrid discretization. Dynamics generated by backward-looking approximations of Euler equations are very different from (163) because the productivity depends on k_t instead of k_{t+1} . Forward-looking dynamics are appropriate to capture the investment decisions that depend on future returns.²⁶

Under the forward-looking approximation $\beta_{t+h}/\beta_t \approx 1/(1 + h\rho_{t+h})$, $\rho_t = \rho$ (that is $\beta_t = \beta_0 e^{-\rho t}$) and $h = 1$, (163) becomes:

$$\frac{\mu_t}{\mu_{t+h}} = \frac{1 + (1 - \tau) \frac{\partial f}{\partial k_{t+1}}(k_{t+1}, g(k_{t+1}))}{1 + \rho}$$

The existence of a steady state requires $\rho_t = \rho$ constant over time. In this case, equations (96)-(97) become:

$$c = (1 - \tau)f(k, g(k)) \quad (166)$$

$$\rho = (1 - \tau) \frac{\partial f}{\partial k}(k, g(k)) \quad (167)$$

(as above, μ_t is stationary at the steady state, while λ_t decreases). Solving (167) for k and replacing in (166) gives c .

Focus now on the steady state of the discretized time model (164)-(165) or, equivalently, when $h = 1$, of the discrete-time model (162)-(163).

Equation (164) evaluated at the steady state gives (166). In addition, we can replace β_{t+h}/β_t by $1/(1 + h\rho_{t+h})$ in (165) to get

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{1 + h(1 - \tau) \frac{\partial f}{\partial k_{t+h}}(k_{t+h}, g(k_{t+h}))}{1 + h\rho_{t+h}}$$

²⁶ As above, the reader must to be aware that the discrete-time Zhang model is only one of the possible first-order discretizations of the continuous-time system (158)-(159) (simply apply equations (141), (142), (143), (144), (145) by replacing $f'(k) - \delta$ with $(1 - \tau) \partial f(k, g(k)) / \partial k$ everywhere).

Immediately, we obtain that $\mu_t = \mu_{t+h}$ and $\rho_{t+h} = \rho$ imply the steady state (167) of the continuous-time model.

The issues of existence and uniqueness rests on the solution of (166)-(167).

Proposition 14 *Let*

$$\varphi(k) \equiv \frac{\partial f}{\partial k}(k, g(k))$$

Under Assumption 7, 8, 9 and boundary conditions

$$\lim_{k \rightarrow 0^+} \varphi(k) < \rho/(1-\tau) \text{ and } \lim_{k \rightarrow +\infty} \varphi(k) > \rho/(1-\tau) \quad (168)$$

or

$$\lim_{k \rightarrow 0^+} \varphi(k) > \rho/(1-\tau) \text{ and } \lim_{k \rightarrow +\infty} \varphi(k) < \rho/(1-\tau) \quad (169)$$

a steady state exists.

Moreover, if, at the steady state, (1) $\varphi'(k) < 0$ in case (168), or (2) $\varphi'(k) > 0$ in case (169), then the steady state is unique.

Proof. Focus first on equation (167): $\varphi(k) = \rho/(1-\tau)$. In order to ensure the existence of a strictly positive k , the boundary conditions (168) and (169), jointly with the continuity of φ , are sufficient.

Derivability of φ is entailed by Assumption 7 ($f(k, g)$ is twice continuously differentiable) and Assumption 9 (derivability of g):

$$\varphi'(k) \equiv \frac{\partial^2 f}{\partial k^2} + \frac{\partial^2 f}{\partial g \partial k} g'(k) \quad (170)$$

Derivability of φ implies continuity.

We notice that, under conditions (168) or (169), and continuity, the number of steady states is odd.

In addition, given a strictly positive k , equation $g = \tau f(k, g)$ has a non-negative solution $g(k)$ because f is continuous, $f(k, 0) \geq 0$ and $\lim_{g \rightarrow +\infty} \partial f / \partial g < 1/\tau$ (this inequality is entailed by Assumption 9). Thus $c = (1-\tau)f(k, g(k))$ is non-negative and μ is strictly positive (Assumption 8).

If there are n steady states k_i with $k_i < k_{i+1}$ and $i = 1, \dots, n$, the sign of φ' changes from steady state k_i to steady state k_{i+1} . In order to ensure the uniqueness, a sufficient condition is that always $\varphi'(k) < 0$ at the steady state in case (168), or always $\varphi'(k) > 0$ at the steady state in case (169). ■

Roughly speaking, (168) and (169) correspond to the cases of dominant increasing and dominant decreasing returns to scale, respectively. As we will see later (equation (178)), $\varphi'(k) > 0$ is a necessary condition to get a Hopf bifurcation. We conclude that increasing returns promotes uniqueness of steady state and occurrence of Hopf bifurcations (and limit cycles). This explains also why, in contrast, the Ramsey-Cass-Koopmans framework is characterized by saddle-path stability.

Focus now on the local dynamics and define the following elasticities:

$$\begin{aligned}(\alpha_1, \alpha_2) &\equiv \left(\frac{\partial f}{\partial k} \frac{k}{f}, \frac{\partial f}{\partial g} \frac{g}{f} \right) \\ (\alpha_{11}, \alpha_{12}) &\equiv \left(\frac{\partial^2 f}{\partial k^2} \frac{k}{\partial f / \partial k}, \frac{\partial^2 f}{\partial g \partial k} \frac{g}{\partial f / \partial k} \right)\end{aligned}$$

Let also

$$(\eta_{11}, \eta_{12}) \equiv \left(\frac{\partial^2 u}{\partial c^2} \frac{c}{\partial u / \partial c}, \frac{\partial^2 u}{\partial g \partial c} \frac{g}{\partial u / \partial c} \right)$$

Notice that $\alpha \equiv \alpha_1$ is the capital share in total income, while $\varepsilon_r \equiv \alpha_{11}$ is the elasticity of the interest rate with respect to the capital intensity with $\varepsilon = -1/\eta_{11} > 0$. Usual assumptions give $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_{11} < 0$, $\alpha_{12} > 0$, $\eta_{11} < 0$, $\eta_{12} \geq 0$.

At the steady state, the discounting is constant over time: $\beta_{t+h}/\beta_t = \beta^h$, where $\beta = e^{-\rho}$. Using $(c, g) = (1 - \tau, \tau) f$, $(\partial f / \partial k, \partial u / \partial c) = (\rho / (1 - \tau), \mu)$ and $c/k = \rho/\alpha_1$, the Jacobian matrix (98) simplifies:

$$J_0 = \begin{bmatrix} \rho \left[1 + \frac{1}{\alpha_1} \left(\alpha_2 - \frac{\partial c}{\partial g} \frac{g}{c} \right) \frac{kg'(k)}{g} \right] & -\frac{\partial c}{\partial \mu} \\ -\rho \left[\alpha_{11} + \alpha_{12} \frac{kg'(k)}{g} \right] \frac{\mu}{k} & 0 \end{bmatrix} \quad (171)$$

Differentiating (157) gives

$$\frac{kg'(k)}{g} = \frac{\alpha_1}{1 - \alpha_2} \quad (172)$$

We observe that Assumption 9 implies $kg'(k)/g > 0$, that is $\alpha_2 < 1$. We get also

$$\left(\frac{dc}{d\mu} \frac{\mu}{c}, \frac{\partial c}{\partial g} \frac{g}{c} \right) = \left(\frac{1}{\eta_{11}}, -\frac{\eta_{12}}{\eta_{11}} \right) \quad (173)$$

Replacing (172) and (173) in (171) gives

$$J_0 = \begin{bmatrix} \rho \frac{1 + \eta_{12}/\eta_{11}}{1 - \alpha_2} & -\frac{1}{\eta_{11}} \frac{\rho}{\alpha_1} \frac{k}{\mu} \\ -\rho \left(\alpha_{11} + \frac{\alpha_1 \alpha_{12}}{1 - \alpha_2} \right) \frac{\mu}{k} & 0 \end{bmatrix} \quad (174)$$

Eventually, (174) becomes:

$$J_0 = \begin{bmatrix} \rho \frac{1 - \varepsilon \eta_{12}}{1 - \alpha_2} & \rho \frac{\varepsilon}{\alpha} \frac{k}{\mu} \\ -\rho \left(\varepsilon_r + \frac{\alpha \alpha_{12}}{1 - \alpha_2} \right) \frac{\mu}{k} & 0 \end{bmatrix}$$

Let us compare the Jacobian matrix of the continuous-time Cass-Koopmans model. Without capital depreciation ($\delta = 0$), $A \equiv \rho\varepsilon/\alpha$, $B \equiv -\rho\varepsilon_r$ and (174) writes

$$J_0 = \begin{bmatrix} \rho & \rho \frac{\varepsilon}{\alpha} \frac{k}{\mu} \\ -\rho \varepsilon_r \frac{\mu}{k} & 0 \end{bmatrix} \quad (175)$$

If there are no externalities in production and utility: $\alpha_2 = \alpha_{12} = \eta_{12} = 0$, the Jacobian of the Cass-Koopmans model eventually reduces to (175). Notice also that setting $\rho = 0$ (no discounting), we recover the Jacobian matrix of the continuous-time Ramsey model with $\delta = 0$.

The trace and the determinant in continuous time (101)-(102) become:

$$\begin{aligned} D_0 &= \varepsilon \rho^2 \left(\frac{\varepsilon_r}{\alpha} + \frac{\alpha_{12}}{1 - \alpha_2} \right) \\ T_0 &= \rho \frac{1 - \varepsilon \eta_{12}}{1 - \alpha_2} \end{aligned}$$

but now, in contrast with the Cass-Koopmans framework ($T_0 > 0$ and $D_0 < 0$) saddle-path stability is no longer ensured and we can have different signs.

In particular, since $\alpha > 0$, $\varepsilon > 0$, $\varepsilon_r < 0$ and $\alpha_2 \in (0, 1)$, we have

$$T_0 \geq 0 \Leftrightarrow \eta_{12} \leq \frac{1}{\varepsilon} \quad (176)$$

$$D_0 > 0 \Leftrightarrow \alpha_{12} > -\frac{1 - \alpha_2}{\alpha} \varepsilon_r \quad (177)$$

As seen above, increasing returns promotes the occurrence of Hopf bifurcations.

We remark also that, using (170) and computing the elasticity of φ at the steady state, we get

$$\frac{k\varphi'(k)}{\varphi(k)} = \varepsilon_r + \frac{\alpha\alpha_{12}}{1 - \alpha_2} \quad (178)$$

Thus, increasing returns ($\varphi'(k) > 0$) require sufficiently large positive externalities ($\alpha_{12} > -(1 - \alpha_2)\varepsilon_r/\alpha$) that imply in turn, according to (177), a necessary condition to the occurrence of Hopf bifurcations ($D_0 > 0$).

Focus now on the hybrid discretization (164)-(165).

At the steady state, (164) becomes $c = (1 - \tau)f$, while, under a forward-looking approximation with a constant ρ ($\beta_{t+h}/\beta_t \approx 1/(1 + h\rho)$), (165) gives $\partial f/\partial k = \rho/(1 - \tau)$. Moreover, the government budget constraint becomes $g = \tau f$. Finally, the derivative of the Lagrangian function with respect to consumption still gives $\partial u/\partial c = \mu$. Thus, unsurprisingly, we recover (166)-(167).

Differentiating (164)-(165) around this steady state or, equivalently, applying (116) with (154) and (155) gives the system $(dk_{t+h}, d\mu_{t+h})^T = J_1 (dk_t, d\mu_t)^T$, where

$$J_1 = \begin{bmatrix} 1 & 0 \\ \frac{h\rho}{1+h\rho} \left[\alpha_{11} + \alpha_{12} \frac{kg'(k)}{g} \right] \frac{\mu}{k} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 + \frac{h\rho}{\alpha_1} \left[\alpha_1 + \left(\alpha_2 - \frac{\partial c}{\partial g} \frac{g}{c} \right) \frac{kg'(k)}{g} \right] & -\frac{h\rho}{\alpha_1} \frac{\partial c}{\partial \mu} \frac{\mu}{c} \frac{k}{\mu} \\ 0 & 1 \end{bmatrix}$$

Replacing (172) and (173) gives

$$J_1 = \begin{bmatrix} 1 & 0 \\ \frac{h\rho}{1+h\rho} \left(\alpha_{11} + \frac{\alpha_1 \alpha_{12}}{1-\alpha_2} \right) \frac{\mu}{k} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 + h\rho \frac{1+\eta_{12}}{1-\alpha_2} & -\frac{1}{\eta_{11}} \frac{h\rho}{\alpha_1} \frac{k}{\mu} \\ 0 & 1 \end{bmatrix}$$

is the capital share in total income, is the elasticity of the interest rate with respect to the capital intensity with > 0 .

Replacing $\alpha \equiv \alpha_1$, $\varepsilon_r \equiv \alpha_{11}$ and $\varepsilon = -1/\eta_{11}$, we get

$$J_1 = \begin{bmatrix} 1 & 0 \\ \frac{h\rho}{1+h\rho} \left(\varepsilon_r + \frac{\alpha \alpha_{12}}{1-\alpha_2} \right) \frac{\mu}{k} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 + h\rho \frac{1-\varepsilon \eta_{12}}{1-\alpha_2} & h\rho \frac{\varepsilon}{\alpha} \frac{k}{\mu} \\ 0 & 1 \end{bmatrix} \quad (179)$$

that is

$$J_1 = \begin{bmatrix} D_1 & \frac{h\rho \varepsilon}{\alpha} \frac{k}{\mu} \\ D_1 (T_1 - D_1 - 1) \frac{\alpha}{h\rho \varepsilon} \frac{\mu}{k} & T_1 - D_1 \end{bmatrix}$$

where

$$D_1 = 1 + h\rho \frac{1 - \varepsilon \eta_{12}}{1 - \alpha_2} \quad (180)$$

$$T_1 = 1 + D_1 - h\rho \frac{\varepsilon}{\alpha} \frac{h\rho}{1+h\rho} \frac{(1-\alpha_2)\varepsilon_r + \alpha \alpha_{12}}{1-\alpha_2} \quad (181)$$

are the determinant and the trace (see also (117)-(118)).

Let us compare the Jacobian matrix of the discretized Cass-Koopmans model (151) with (179). Without capital depreciation ($\delta = 0$), $A \equiv \rho\varepsilon/\alpha$, $B \equiv -\rho\varepsilon_r$ and (151) simplifies to

$$J_1 \equiv \begin{bmatrix} 1 + h\rho & h\rho \frac{\varepsilon}{\alpha} \frac{k}{\mu} \\ -h\rho \varepsilon_r \frac{\mu}{k} & 1 - h\rho \frac{\varepsilon}{\alpha} \frac{h\rho}{1+h\rho} \varepsilon_r \end{bmatrix} \quad (182)$$

If there are no externalities in production and utility: $\alpha_2 = \alpha_{12} = \eta_{12} = 0$ and (179) eventually reduces also to (182). Notice also that setting $\rho = 0$ (no discounting), we recover the Jacobian matrix of the continuous-time Ramsey model with $\delta = 0$.

The main interest of the Zhang model (2000) is the occurrence of a Hopf bifurcation.

Hopf bifurcation in continuous time generically requires: $\alpha = 0$ and $\beta \neq 0$, that is

$$\begin{aligned} T_0 &= 0 \\ D_0 &> T_0^2/4 = 0 \end{aligned}$$

According to (176) and (177), $T_0 = 0$ and $D_0 > 0$ are equivalent to

$$\eta_{12} = 1/\varepsilon (> 0) \quad (183)$$

$$\alpha_{12} > -\frac{1-\alpha_2}{\alpha} \varepsilon_r (> 0) \quad (184)$$

respectively. In other terms, either the externalities in the production function ($\eta_{12} > 0$) or in the utility function ($\alpha_{12} > 0$) are crucial in order to find limit cycles. Notice also that both the externalities are necessary. For instance, in the Barro model (1990), even if $\eta_{12} > 0$, saddle-path stability prevails because $\alpha_{12} = 0$.

It is known that a Hopf bifurcation generically arises in discrete time if and only if $D_1 = 1$ and $T_1^2 \leq 4$ (see the section on the Hopf equivalence).

Replacing (180) in $D_1 = 1$, we get

$$\eta_{12} = 1/\varepsilon \quad (185)$$

as in the continuous-time case (whatever the discretization step), while, replacing (181) in $T_1^2 \leq 4$ and using $\eta_{12} = 1/\varepsilon$, we need

$$\left(2 - h\rho \frac{\varepsilon}{\alpha} \frac{h\rho}{1+h\rho} \frac{(1-\alpha_2)\varepsilon_r + \alpha\alpha_{12}}{1-\alpha_2}\right)^2 \leq 4$$

that is,

$$\alpha_{12} \geq -\frac{1-\alpha_2}{\alpha}\varepsilon_r \quad (186)$$

and

$$\frac{(h\rho)^2}{1+h\rho} \leq 4 \frac{\alpha}{\varepsilon} \frac{1-\alpha_2}{(1-\alpha_2)\varepsilon_r + \alpha\alpha_{12}} \quad (187)$$

Let

$$\omega \equiv 2 \frac{\alpha}{\varepsilon} \frac{1-\alpha_2}{(1-\alpha_2)\varepsilon_r + \alpha\alpha_{12}} (> 0)$$

(187) is equivalent to

$$h \leq h^* \equiv \frac{1}{\rho} \left[\omega + \sqrt{\omega^2 + 2\omega} \right] \left(> \frac{\omega}{\rho} > 0 \right)$$

Conditions (185) and (186) are respectively equivalent to conditions (183) and (184). Since the RHS of (187) is positive under condition (186), inequality (187) is satisfied for $h < h^*$. In other terms, as proved in the section on the Hopf equivalence, under a sufficiently small discretization step, a Hopf bifurcation occurs in discrete time if and only if it arises in continuous-time.

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